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The Local Potential Approximation of the Renormalization Group

by

Christopher Simon Francis Harvey-Fros

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This thesis was submitted for examination in March 1999. It does not necessarily represent the final form of the thesis as deposited in the University after examination.

Dedicated to my family

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ABSTRACT

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We introduce Wilson's, or Polchinski's, exact renormalization group, and review the Local Potential Approximation as applied to scalar field theory. Focusing on the Polchinski flow equation, standard methods are investigated, and by choosing restrictions to some sub-manifold of coupling constant space we arrive at a very promising variational approximation method. Within the Local Potential Approximation, we construct a function, C, of the coupling constants; it has the property that (for unitary theories) it decreases monotonically along flows and is stationary only at fixed points - where it 'counts degrees of freedom', *i.e.* is extensive, counting one for each Gaussian scalar.

In the latter part of the thesis, the Local Potential Approximation is used to derive a non-trivial Polchinski flow equation to include Fermi fields. Our flow equation does not support chirally invariant solutions and does not reproduce the features associated with the corresponding invariant theories. We solve both for a finite number of components, N, and within the large N limit. The Legendre flow equation provides a comparison with exact results in the large N limit. In this limit, it is solved to yield both chirally invariant and non-invariant solutions.

Contents

List of Figures

List of Tables

Preface

Original work appears in the latter parts of chapters two and three and throughout chapter four. Some of the work presented has been published in

J.Generowicz, C.Harvey-Fros and T.R. Morris, Phys. Lett. **B407**, (1997) 27.

Other work awaits publication.

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Chapter 1

Introduction

1.1 Motivation and Structure

The purist aspires to a reliable method of performing non-perturbative calculations in quantum field theory whereas the practically minded physicist desires an efficient cost effective procedure of accurately generating physical quantities. We will argue that the methods presented here offer us hope of attaining these desires and aspirations. The strength of the scheme discussed exhibits itself in two important features. First the method can be improved in a systematic way producing a sequence of converging approximations and second, there is no small parameter to control the approximation (unlike perturbation theory), allowing us to reach and test new and interesting areas of physics. In particular we demonstrate that the method produces competitive results in the context of scalar field theory, both numerically and theoretically via critical exponents and Zamalodchikov's C-theorem respectively. Many authors use these methods for alternative reasons, *e.g.* to calculate effective potentials. The successes presented here motivate our attempts to extend the formalism to include Fermions.

We begin the thesis by introducing the essential concepts of the renormalization group methods from a field theory perspective. The inevitable parallels with theories of critical phenomena are drawn with the usual lengthy and detailed discussions left to reference [1-5]. Then the methods applied to scalar field theory are reviewed and compared, paying particular attention to a Z_2-invariant one component field theory in three dimensions. We develop a new variational approach which is applied to the scalar field theory under consideration and compared to the more standard techniques. We give a brief introduction to conformal field theory and review Zamalodchikov's C-theorem. Then, completing the defence of the Local Potential Approximation, we show how an appropriate generalisation of Zamalodchikov's C-function can be found within this scheme. The thesis finishes with an outline of progress made and problems associated with attempts to extend the work beyond purely scalar field theory. In particular we compare the Local Potential Approximation in the context of Fermionic field theories with exact results obtained in the large N limit. This leads us to an understanding of the corresponding results at finite N. We will work in Euclidean space throughout, unless stated otherwise.

1.2 Renormalization

Prior to the main discussion of the thesis we wish to stress the modern point of view of renormalization as distinct to that presented in older textbooks. Here, we define renormalization as the procedure of re-expressing the parameters which define a problem in terms of some other, perhaps simpler set, while keeping unchanged those physical aspects of the problem which are of interest [4].

1.3 The Ising model

A classical theory of spin-spin coupling in a ferromagnet can be described by the Hamiltonian,

$$H(s,h) = -\sum_{n,m} V_{n,m} s_n . s_m + h \sum_n s_n \tag{1.1}$$

where h is the external magnetic field and $V_{n,m}$ is the coupling of spins s_n and s_m at sites x_n and x_m respectively. Neighbouring sites in the lattice of spins are separated by a distance a. If we assume the lattice is D-dimensional hyper-cubic and we only have nearest neighbour interactions, we can write

$$-\sum_{n,m} V_{n,m} s_n . s_m = K \sum_n \left(\sum_\mu (s_{n+\mu} - s_n)^2 - 2D s_n^2 \right) \tag{1.2}$$

where $\mu = 1, \cdots, D$. Within the Ising model s is a scalar which takes the values $s = \pm 1$. An effective[1] description of the partition function is given by

$$Z(h, \beta) = \int \prod_n ds_n \rho(s_n) e^{-\beta H(s,h)} \tag{1.3}$$

where $\beta = \frac{1}{K_B T}$ and $\rho(s_n)$ is a weight factor describing the averaged local, microscopic properties of the spin. A choice like

$$\rho(s_n) \propto e^{-cs_n^2 - \lambda s_n^4} \tag{1.4}$$

gives a convenient description. The partition function (1.3) may now be written as

$$Z(K, \nu, \lambda, h) = \int \prod_n ds_n e^{-H(s;K,\nu,\lambda,h)} \tag{1.5}$$

where the effective Hamiltonian is given by

$$H(s; K, \nu, \lambda, h) = \sum_n \left(K(\beta) \sum_\mu (s_{n+\mu} - s_n)^2 + \nu(\beta) s_n^2 + \lambda(\beta) s_n^4 + h s_n \right) \tag{1.6}$$

and $\nu(\beta) = c(\beta) - 2D$. Consider the potential $\nu(\beta)s^2 + \lambda(\beta)s^4$ (figure 1.1). A ferromagnetic transition (symmetry breaking) occurs at $\nu(\beta_c) = 0$. The value of T, for which $\nu(\beta_c) = 0$, is called the critical temperature, T_c.

Minimising the effective Hamiltonian in (1.6) with $\nu(\beta) > 0$ leads to a ground state where all the $s_n = 0$, leading to zero magnetisation,

$$< s > \equiv \frac{1}{V} \sum_n s_n = 0. \tag{1.7}$$

However the ground state for $\nu(\beta) < 0$ corresponds to all s_n aligned with $s_n = \sqrt{\frac{-\nu}{2\lambda}}$, leading to magnetisation,

$$< s > = \sqrt{\frac{-\nu}{2\lambda}}. \tag{1.8}$$

The Ginzburg Landau theory of ferromagnetic transitions assumes that $K(\beta)$, $\nu(\beta)$ and $\lambda(\beta)$

[1] we replace the discrete spins by continuous variables which by universality will produce the critical behaviour of the Ising model

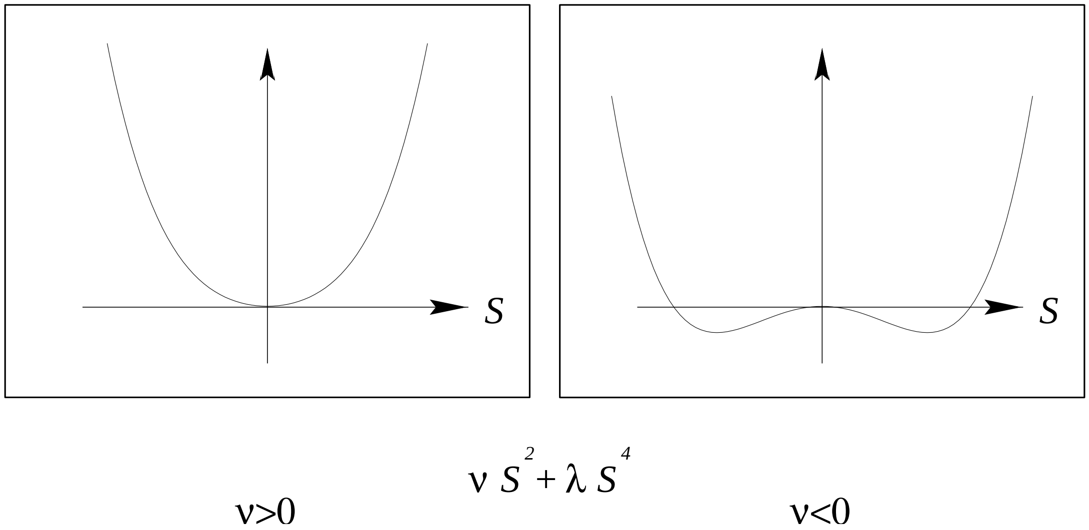

Figure 1.1: Symmetry breaking

are smooth functions of temperature. Thus we might write

$$\nu(\beta) \propto (\beta - \beta_c) \tag{1.9}$$

for $\beta \approx \beta_c$, leading to

$$< s > \propto \sqrt{\beta - \beta_c} \tag{1.10}$$

for $\beta > \beta_c$. This shows the typical non-analytic behaviour at a phase transition, as indicated at $\beta = \beta_c$ in figure 1.2. Near the critical point, translational invariance implies that the correlation function will behave as

$$< (s_n - < s >)(s_m - < s >) > \sim |x_n - x_m|^{-(D-2+\eta)} \tag{1.11}$$

for the range $a \ll |x_n - x_m| \ll \xi(h, \beta)$. The lack of any dependence on a fundamental length scale reflects that at criticality the system is scale invariant (to be discussed in greater depth later). The correlation length, ξ, can be thought of as the size to which we can reduce the system without qualitatively changing its properties. The anomalous dimension, η, is effectively defined by (1.11). We will discuss this quantity in greater detail later.

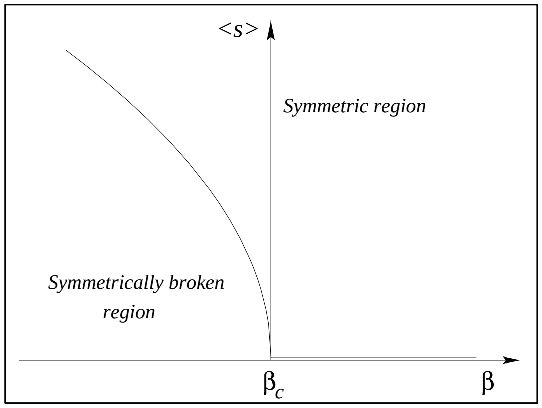

Figure 1.2: Magnetisation as a function of temperature for the Ising model with zero external magnetic field

1.4 Lattice Field theory

Consider the Lagrangian for one component ϕ^4 scalar field theory in D-dimensional Euclidean space,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{1.12}$$

We choose to regularise the field theory by discretising spacetime using a finite volume hypercubic lattice. This leads to a direct comparison with the Ising model of a ferromagnet. We let

$$x \to x_n \equiv an\hat{e}_\mu, \tag{1.13}$$

$$\phi(x) \to \phi_n \equiv \phi(x_n), \tag{1.14}$$

$$\partial_\mu\phi \to \frac{1}{a}\left(\phi(x_n + \hat{e}_\mu) - \phi(x_n)\right) \equiv \frac{1}{a}\left(\phi_{n+\mu} - \phi_n\right) \tag{1.15}$$

and the measure

$$\mathcal{D}\phi \to \prod_n^V d\phi_n \tag{1.16}$$

where $\hat{e}_\mu$ denotes D orthonormal vectors and a is the lattice spacing.

By scaling the fields, sources and coupling constants, such that they are dimensionless,

$$\lambda = g^2 a^{D-4}, \quad \phi' = g a^{\frac{1}{2}(D-2)}\phi, \quad J' = g a^{\frac{1}{2}(D+2)}J, \tag{1.17}$$

the discretised action can be written

$$S(\phi', J') = \sum_n \left(\frac{1}{2} \sum_\mu (\phi'_{n+\mu} - \phi'_n)^2 + \frac{1}{2} m^2 a^2 \phi'^2_n + \frac{1}{4!} \phi'^4_n + J'_n \phi'_n \right). \tag{1.18}$$

If we compare this with the Hamiltonian for the spin system (1.6) we see that at the classical (or Ginzburg Landau) level the continuum limit of the dimensionless lattice field theory ($a \to 0$) corresponds to approaching the critical point of the ferromagnetic transition ($\nu(\beta) \to 0$) because of the identification

$$m^2 a^2 \sim \nu(\beta). \tag{1.19}$$

Furthermore, we associate the even symmetry of the lattice field theory $(\phi \to -\phi)$[2] with a 'flipped spin' symmetry of the spin system ($s \to -s$). Thus the continuum field theory can be viewed as a critical classical spin system.

1.5 Universality

The association between the lattice ϕ^4 theory and the Ising model may appear contrived, however it is found that different physical systems form 'universality classes', in the sense that quantum field theories can be used to describe critical phenomena for a range of models. Although there is no proof, there is considerable evidence that $O(N)$ symmetric scalar field theories correspond to various universality classes according to the value of N [1];

<u>N=0:</u> *Critical behaviour of polymers.* This is only formally defined in the limit $N \to 0$ as noted by Gennes [6].

<u>N=1:</u> *Liquid-vapour transition and alloy order-disorder transition.* Observe that we have no internal symmetries and the liquid-vapour transition can be modelled by particles living on a lattice, allowing the occupation of each site to be either zero or unity (equivalent to the Ising model) [7].

<u>N=2:</u> *He^2 superfluid phase transition and planar ferromagnets.* The planar ferromagnet has symmetry in a plane corresponding to an $O(2)$ invariant scalar field theory.

[2] later we will be interested in a Z_2-invariant one component scalar field theory

N=3: *Ferromagnetic phase transitions.* Here we observe a three dimensional symmetry corresponding to an $O(3)$ invariant field theory.

N=4: *Chiral phase transition.* It has been postulated to correspond to the chiral phase transition for two flavours of quarks.

Arguments presented in the following analysis provide a phenomenological explanation of universality.

1.6 Kadanoff blocking

We now drop the distinction between the Ising model and the lattice field theory and use a renormalization procedure known as Kadanoff blocking [8] to reduce the number of degrees of freedom. Our toy model Hamiltonian will be

$$H = \sum_n \left(\sum_\mu \frac{1}{2}(\phi_{n+\mu} - \phi_n)^2 + \mu\phi_n^2 + \lambda\phi_n^4 \right). \tag{1.20}$$

We begin by dividing the original lattice into blocks of size s^D where s is an integer, and define an average field in each block $B_s(n')$ labelled n' (figure 1.3) by

$$\phi'_{n'} = \frac{1}{s^D} \sum_{n \epsilon B_s(n')} \phi_n. \tag{1.21}$$

We end by scaling the blocks back to the original size

$$x_s = \frac{x}{s} \tag{1.22}$$

and

$$\phi_s = s^{\frac{1}{2}(D-2+\eta)} \phi' \tag{1.23}$$

to ensure that the short distance properties of the correlation functions (physics) are left unchanged. This can be seen by considering the effect of the block transformation on the correlation functions:

$$< \phi'_{n'}\phi'_{0'} > \;=\; \frac{1}{s^{2D}} < \sum_{n\epsilon B_s(n')} \phi_n \sum_{m\epsilon B_s(0')} \phi_m > \tag{1.24}$$

$$\approx \; < \phi_{sn'}\phi_0 > \tag{1.25}$$

since all the s^{2D} correlation functions in (1.24) are at essentially the same distance if n' is very large. Then, from (1.11) we see that

$$< \phi_n\phi_0 > \sim \frac{1}{n^{D-2+\eta}} \tag{1.26}$$

and thus for the correlation functions to be left unchanged by blocking we must simultaneously scale the fields as indicated by (1.23).

The new Hamiltonian, $H_s(\phi_s)$, will not be identical to the original, (1.20), but will contain infinitely many additional arbitrarily complicated terms including for example

$$(\phi_{n+\mu} - \phi_n)^2\phi_n^2, \qquad \phi_{n+\mu}^2\phi_n^4 \quad \text{and} \quad (\phi_{n+\mu} - 2\phi_n + \phi_{n+\mu})^2. \tag{1.27}$$

Thus, given that we will consider successive blockings, it is natural to start with a completely general action including all terms which obey the symmetries of the theory under consideration. In general these terms will be infinite in number. Then, each blocking can be viewed as a mapping of the 'coupling constant space' onto itself, called the renormalization group transformation, denoted $T(s)$. We can see the group structure explicity through $T(s_1)T(s_2) = T(s_1 s_2)$, although strictly speaking we only have a semi-group, since there will not necessarily be an inverse transformation.

In terms of the ferromagnet we imagine looking at a sample through a microscope such that our eyes can see spin variations down to a certain size. Then Kadanoff blocking represents the operation of decreasing the magnification of the microscope by the factor s.

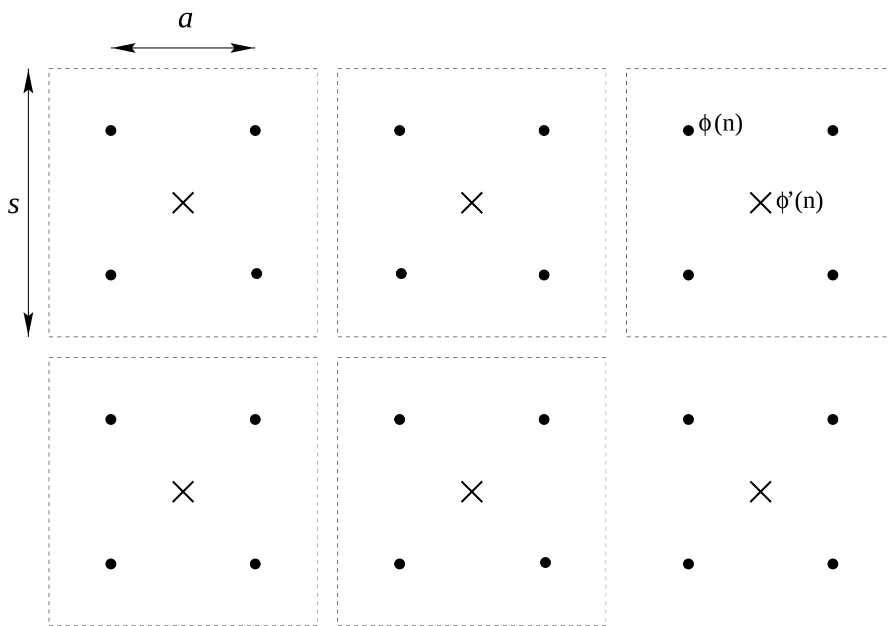

Figure 1.3: Kadanoff Blocking

1.7 Effective Lagrangians

The lattice we discussed earlier is simply a method of regularisation which corresponds (more or less) to introducing an overall momentum cutoff ($\Lambda_o \sim \frac{1}{a}$). Thus, Kadanoff blocking can be interpreted as integrating out high momentum modes[3] and scaling the cutoff back to its original size, $\Lambda = \Lambda_o$, such that the physics remains unchanged. However, it is simpler and equivalent to ensure that all variables are 'measured' in units of Λ, *i.e.* we change variables to ones that are dimensionless, by dividing by Λ raised to the power of their scaling dimensions. Then the action of lowering Λ reproduces the scaling steps in (1.22) and (1.23).

Here we are considering directly a continuum effective Lagrangian describing physics over a limited range of energies, *i.e.* momentum less than Λ_o. The most famous example of an effective theory is the Standard Model which produces good results below the scale of Grand Unification, where new physics occurs. Throughout this thesis we will be dealing primarily with the Legendre and Wilsonian (or Polchinski) effective actions. Begin by considering a general effective action, S^{eff}, which we assume to be valid below the overall momentum cutoff, Λ_o. Below Λ_o the effective Lagrangian will be described by an infinite set of couplings. Then, to

[3]introducing a secondary momentum cutoff, Λ

look at physics at some energy scale $E \ll \Lambda_o$ we lower the cutoff to a scale $\Lambda \sim E$ and allow the couplings to flow in order to keep the physics constant (partition function invariant with respect to Λ). Then the action will evolve according to a flow equation of the general form

$$\Lambda \frac{\partial S^{eff}}{\partial \Lambda} = f[S^{eff}]. \tag{1.28}$$

Conventionally we parametrise flow equations in terms of the 'renormalization time', defined by

$$t = \ln \left(\frac{\Lambda_o}{\Lambda} \right) \tag{1.29}$$

so that t runs from 0 to ∞ as Λ runs from Λ_o to 0. Generally we will be interested in solving in the continuum limit, corresponding to $\Lambda_o \to \infty$, where we require the solutions of interest to be insensitive to Λ_o.

The effective theory is said to be renormalizable if we can calculate physical processes once we have determined a finite number of couplings, known as relevant. Relevant couplings , denoted g^R, flow away from some 'fixed point' value, g^R_*. The other, irrelevant couplings, g^I, flow toward some 'fixed point' value, g^I_*. The renormalization (semi-) group moves the action along the trajectories in the coupling space.

1.8 Fixed points and Perturbations

If the bare action (at $\Lambda = \Lambda_o$) is chosen in the neighbourhood of the fixed point in such a way that the relevant couplings are their fixed point values ($g^R_o = g^R_*$) and the irrelevant couplings are close to their fixed point values then the action will flow into that fixed point, S^{eff}_*, defined by

$$\Lambda \frac{\partial S^{eff}_*}{\partial \Lambda} = f[S^{eff}_*] = 0. \tag{1.30}$$

A fixed point is associated with a critical surface in the coupling space which contains all the bare actions which flow into that fixed point. Figure 1.4 illustrates the neighbourhood of a fixed point with the critical surface and renormalization group trajectory associated with it. In practice this space is infinite dimensional. Similarly, if we begin with the ferromagnet at its critical point, $T = T_c$, and decrease the magnification sufficiently by Kadanoff blocking, eventually we shall observe no further change (*i.e.* we reach a fixed point).

We enquire about the stability of the fixed points by perturbing about the fixed point solution. For example, later we will expand a dimensionless potential (for dimensionless scalar fields) as

$$V(\phi, t) = V_*(\phi) + \epsilon \sum_n f_n(\phi) e^{\lambda_n t} \tag{1.31}$$

where V_* is the fixed point solution under consideration and the perturbation is considered for small ϵ. We immediately notice that $\lambda_n > 0$ are associated with relevant directions in the coupling space, in which the perturbations grow with t and similarly $\lambda_n < 0$ are associated with irrelevant directions. Fixed points are characterised by the λ_n associated with them. If $\lambda_n = 0$ we can to first order add any amount of the corresponding 'marginal perturbation' to the fixed point without introducing any t dependence. Such a scenario can result in lines of fixed points, and does in a few specific cases to be discussed.

Substituting (1.29) into (1.31) we find that the dimensionless perturbation around a fixed point is given by

$$\delta V = \epsilon \sum_n f_n(\phi) \left(\frac{\Lambda_o}{\Lambda}\right)^{\lambda_n}. \tag{1.32}$$

However, more generally the flow equation (1.28) implies that we can write

$$\Lambda \frac{\partial}{\partial \Lambda} \delta V_\Lambda = -L(\delta V_\Lambda) \tag{1.33}$$

where L is a linear operator acting on δV_Λ. Thus from (1.32) we see that the λ_n form a discrete set of eigenvalues for L, where the $f_n(\phi)$ are the corresponding eigenfunctions.

Since we exchange all dimensionful variables for dimensionless ones using Λ, independence of Λ would imply lack of dependence on any scale at all. Thus a fixed point, defined by (1.30), corresponds to a scale invariant theory and thus is associated with either divergent or vanishing correlation lengths. Hence a fixed point is massless (or infinitely massive) respectively. A massive theory leads to no propagation and is thus considered less interesting[4]. Later, we will discover a trivial fixed point known as the Gaussian which represents a massless quantum field theory. The value of the coefficient of ϕ^2 will generally be non-zero at a massless fixed

[4]such a fixed point is known as non-critical or high temperature

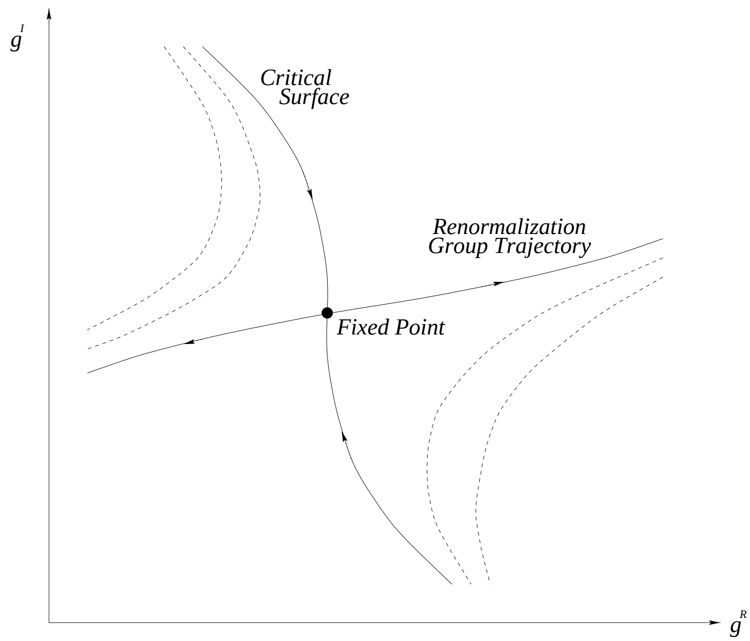

Figure 1.4: A two dimensional projection of the flow near a fixed point

point (with the exception of the Gaussian), but will be precisely the required 'counter-term' to cancel the remaining quantum corrections, leaving the theory overall massless. Furthermore, the massless fixed point must correspond to the continuum limit ($\Lambda_0 \to \infty$), for otherwise Λ_o would set the scale.

Now we are in a position to make the concept of universality more concrete. Universality classes are characterised by their eigenvalues, λ_n, (and more generally by the dimensions of the operator spectrum and their symmetries) and thus the eigenvalues calculated from field theory may, at least in principle, be compared with measurements taken in the laboratory. We interpret universality as arising due to the fact that near the critical surface the correlation length becomes large. As a result the behaviour of the system under consideration is to a large extent unaffected by the detailed structure of the model under consideration [9]. Thus models 'converge' in the critical domain leading to universality.

1.9 Critical exponents

Within the Local Potential Approximation we will use the renormalization group to compute experimentally measurable critical exponents, which are related to the λ_n. Specifically, we define for the largest positive λ

$$\nu = \frac{1}{\lambda}, \tag{1.34}$$

and for the negative λ closest to zero

$$\omega = |\lambda|. \tag{1.35}$$

To calculate these critical exponents we need to consider the behaviour near the fixed point. Suppose we are near a fixed point but not on the critical surface. The flow of the action will be dominated by the largest relevant eigenvalue, and thus by (1.32) we can write

$$\delta V_\Lambda \sim \left(\frac{\Lambda_o}{\Lambda}\right)^{\lambda_L} f_L(\phi) \tag{1.36}$$

where λ_L is the largest eigenvalue and $f_L(\phi)$ is the eigenperturbation associated with it. We see that $f_L(\phi)$ has scaling dimension $-\lambda_L$ and is associated with a coupling of dimension λ_L[5]. Now consider the situation at $\Lambda = \Lambda_o$. There will be an eigenperturbation corresponding to the deviation from criticality. This will be the dominant eigenperturbation determining the statistical state of the system. Therefore, we associate this operator with $f_L(\phi)$:

$$\delta V_{\Lambda_o} = (T - T_c) f_L(\phi) + O((T - T_c)^2). \tag{1.37}$$

As we know the coefficient of $f_L(\phi)$ has scaling dimension λ_L, we see that the scaling of the temperature difference is also λ_L. Therefore as the correlation length ξ has scaling dimension -1, we must have that

$$\xi \sim (T - T_c)^{-\frac{1}{\lambda_L}} \tag{1.38}$$

$$= (T - T_c)^{-\nu}. \tag{1.39}$$

Corrections to this scaling behaviour are given by the subleading exponents, and are dominated

[5] recall that Λ has been used to trade dimensionful couplings for dimensionless ones

by ω. Specifically

$$\xi \sim (T - T_c)^{-\nu} + a_\xi (T - T_c)^{(\omega - 1)\nu}. \tag{1.40}$$

Strictly speaking this is only pertinent to a critical point with one relevant direction. With more than one relevant eigenvalue one normally considers a universal scaling function of the relevant parameters and then again the least negative eigenvalue quantifies the first corrections to scaling.

1.10 β-functions

Let us remark that the flow equation in (1.28) can be rewritten in terms of a complete set of parameters g^i (*a.k.a.* the coupling constants). In general (away from a continuum limit) this will require an infinite number of couplings. Then (1.28) may be rewritten in terms of the β-functions defined as

$$\beta^i(g) = \frac{dg^i}{dt}. \tag{1.41}$$

By the definition (1.30), the β-functions vanish at a fixed point where $g^i = g^i_*$ for all i. Later we will perform this reformulation within an approximation.

1.11 Anomalous dimension

As described it is convenient to rewrite the flow equations in terms of dimensionless quantities, denoted here with a subscript *dim*. We do this by scaling the couplings and fields by appropriate powers of Λ. Hence, for classical scalar field theory we would let

$$\phi = \phi_{dim} \Lambda^{\frac{1}{2}(D-2)} \tag{1.42}$$

such that the kinetic term in the action is overall dimensionless. However, to take account of wave function renormalization in the corresponding quantum theory we use the anomalous dimension, η, by letting

$$\phi = \phi_{dim} \Lambda^{\frac{1}{2}(D-2+\eta)}. \tag{1.43}$$

Compare this with (1.23) to deduce that this is indeed the anomalous dimension introduced by (1.11). Likewise we scale the other operators (ϕ^{2i} for integer $i = 0, 1, \cdots$)[6] using a set of parameters which we denote γ_i which may be deduced from the renormalization group (for example $\phi^2 \sim \phi^2_{dim}\Lambda^{D-2+\gamma_1}$). Similarly the couplings, g^i, scale in an γ_i dependent way leaving the action dimensionless. Then considering the scaling behaviour of the couplings with Λ we deduce which correspond to relevant operators and irrelevant operators. For example, the mass is found to scale as

$$m^2 \sim m^2_{dim}\Lambda^{2-\gamma_1} \tag{1.44}$$

which is clearly relevant provided $\gamma_1 < 2$. In fact, in this way it may be possible for classically relevant operators to become irrelevant and visa-versa when quantum corrections are included. From (1.44) it is clear that a fixed point with $\gamma_1 < 2$ corresponds to choices of couplings and interactions in the Lagrangian such that it produces a theory which is massless, regardless of the value of the dimensionless mass[7] (m^2_{dim}), whereas $\gamma_1 > 2$ corresponds to an infinitely massive theory, and $\gamma_1 = 2$ is marginal.

1.12 The Momentum expansion

In practice, exact flow equations prove very difficult to handle. Here we will be interested in using the so called Local Potential Approximation (LPA hereafter) as a method of deriving flow equations which can be solved with relative ease. We consider the effective Lagrangian as an expansion in powers of momentum and within the LPA truncate at the zeroth order, hence discarding all momentum dependence. For example, for scalar field theory we could write

$$S^{eff} = \int d^D x \left(V(\phi, t) + \frac{1}{2}(\partial_\mu \phi)^2 \right) \tag{1.45}$$

in D dimensions. This approximation has been extensively investigated [10-13] and has produced some competitive results, as will be discussed. Clearly as all quantum corrections to the kinetic term are neglected, the LPA is equivalent to assuming vanishing anomalous dimension.

[6]here we are primarily interested in a Z_2-invariant theory and concentrate on terms which maintain this symmetry

[7]recall that at a fixed point, the dimensionless mass m^2_{dim} is fixed under variations in Λ

By including successive momentum dependent terms it is expected that a sequence of converging improvements over the LPA will be generated [14] allowing for non-zero anomalous dimension. For example at first order we may paramaterise S^{eff} by

$$S^{eff} = \int d^D x \left(V(\phi, t) + \frac{1}{2} K(\phi, t)(\partial_\mu \phi^a)^2 + \frac{1}{2} Z(\phi, t)(\phi^a \partial_\mu \phi^a)^2 \right) \tag{1.46}$$

where $a = 1, \cdots, N$ for an N-component theory. This leads to three partial differential flow equations for V, K and Z. For one component field theory we can simplify the effective action by eliminating K and Z in favour of a single quantity, $K + \phi^2 Z$ [14]. We should note, that in general the expansion (beyond LPA) is found to destroy 'reparameterisation symmetry' [15], the requirement that physics should not depend on the normalisation of ϕ (*i.e.* $\phi \to \Omega\phi$ and $J \to \frac{J}{\Omega}$). In some cases this problem can be circumvented by a careful choice of cutoff.

1.13 Discussion

The title 'Renormalization Group' is somewhat unfortunate because it may be confused with the perturbative (Callen Symanzik) renormalization group that appears in most textbook treatments of quantum field theory. It should be stressed that more generally the renormalization group is merely a framework, a set of ideas which may also be applied to problems quite unrelated to field theory. Thus, Kadanoff blocking simply represents the traditional introduction to renormalization from a field theory perspective. However, whatever the motivation, these methods have the common feature that they lead to mathematical equations describing renormalization group flows in some complicated parameter space. It is the study of these flows, and what information they yield about the system, which is the essence of renormalization group theory. With quantum field theory, the renormalization group methods become particularly advantageous when we consider the continuum limit. We have found that a fixed point of the flow equation corresponds to the continuum limit. Thus, by solving for a fixed point we are solving directly in the continuum.

In this chapter and throughout the rest of this thesis we will be primarily interested in calculating critical exponents, which provide a convenient testing ground for the Local Potential Approximation. However, the ultimate motivation for a fixed point search runs much deeper.

To make a quantum field theory well defined, traditionally we impose an ultra-violet cutoff, Λ_o. Then, the bare parameters in the Lagrangian are chosen functions of Λ_o, such that as the cutoff is removed, we obtain a theory with only finite mass scales. On the natural scale Λ_o this means that all the mass scales must vanish as $\Lambda_o \to \infty$. This in turn implies that the choice of bare parameters must be such that it results in a fixed point (as discussed in section 1.8). The Standard Model of particle physics is based entirely around the Gaussian fixed point, which has problems. The scalar fields of the Higgs sector may define a trivial theory about the Gaussian fixed point, in the sense that as Λ_o is removed (*i.e.* $\Lambda_o \to \infty$), all choices of bare couplings result in a free theory. Also, there has been no explanation for why the masses and couplings take the values observed experimentally. One possibility is that the Standard Model might be defined about a non-trivial fixed point. Such a fixed point would have couplings at some non-zero values and would be an interacting scale free theory (conformal field theory).

Chapter 2

Scalar field theory

In the following analysis we make more concrete some of the appealing arguments and concepts presented in chapter one. As discussed, the ultimate goal is to derive and solve flow equations for non-trivial theories. However, it is useful to begin with the simplest scalar field theory to develop techniques and understanding. As indicated, we will use critical exponents as a testing ground for the approximation scheme under consideration. In the first part of this chapter, we investigate standard methods (*i.e.* truncations and shooting) as applied to a Z_2-invariant one component field theory. In particular we will be interested in the non-trivial Wilson-Fisher fixed point found in three dimensions, however some analysis between two and four dimensions is also presented. Subsequently, we develop a new variational procedure and compare with the standard methods presented. The large N limit of the Local Potential Approximation is solved analytically and found to yield the eigenvalue spectrum exactly. In the closing section we compare these results with those due to other leading methods.

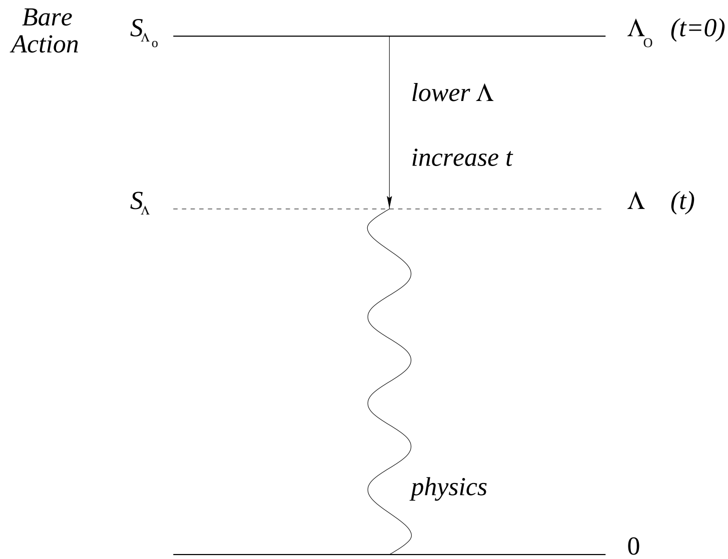

Figure 2.1: A schematic representation of the momentum cutoffs used

2.1 The Wilsonian effective action

We begin by making some more precise statements about the Wilsonian effective action [16] discussed in section 1.7. The generating functional

$$Z = \int_0^{\Lambda_o} \mathcal{D}\phi \; e^{-S_{bare}} \tag{2.1}$$

is replaced with

$$Z = \int_0^{\Lambda} \mathcal{D}\phi \; e^{-S_{ren}}. \tag{2.2}$$

(In practice, we ignore a multiplicative Λ-dependent factor which does not contribute in any correlation function). We are then free to lower Λ as represented schematically in figure 2.1.

Consider an N-component scalar field theory with an ultra-violet (UV hereafter) cutoff imposed using the kinetic term. We write the Λ-dependent action as

$$S_{ren} = \frac{1}{2}\phi_a.\Delta^{-1}_{UV}.\phi_a + S_\Lambda \tag{2.3}$$

where the dots have the usual interpretation as representing an integral over position space[1]

[1] *e.g.* $f.g = f_x g_{xy} = \int d^D x f(x) g(x,y)$ and $\mathrm{tr}(g) = g_{xx} = \int d^D x g(x,x)$

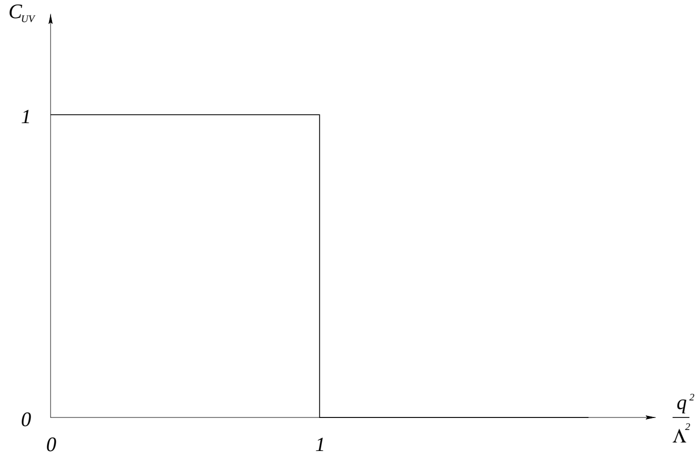

Figure 2.2: An ultra-violet cutoff function

and where a runs from 1 to N, in momentum space $\Delta_{UV} = \frac{C_{UV}}{q^2}$ and $C_{UV}(\frac{q^2}{\Lambda^2})$ is an UV cutoff function. We insist C_{UV} is a function of $\frac{q^2}{\Lambda^2}$ ensuring that it is dimensionless and Lorentz invariant, and that it falls off sufficiently rapidly such that the theory is efficiently regulated. Furthermore, we require that C_{UV} is a profile that acts as a UV cutoff, *i.e.* $C_{UV}(0) = 1$ and $C_{UV} \to 0$ for $q \to \infty$. It might take the form shown in figure 2.2, however it sometimes proves convenient to choose C_{UV} as a smooth [14] (*e.g.*, power law) function. Note that as $C_{UV} \to 0$ for $q > \Lambda$ in figure 2.2, S_{ren} in (2.3) diverges providing the required cutoff for the partition function in (2.2). Similarly we introduce an infra-red (IR hereafter) cutoff by modifying the propagator $\frac{1}{q^2}$ to $\Delta_{IR} = \frac{C_{IR}}{q^2}$ where $C_{IR}(\frac{q^2}{\Lambda^2})$ is a profile with the properties $C_{IR}(0) = 0$ and $C_{IR} \to 1$ as $q \to \infty$. We require that the two cutoffs are related by

$$C_{IR}\left(\frac{q^2}{\Lambda^2}\right) + C_{UV}\left(\frac{q^2}{\Lambda^2}\right) = 1. \tag{2.4}$$

No source terms have been included in the generating functionals, (2.1) and (2.2). If a source term, normally denoted J, is included in the bare action, then the renormalization procedure will result in additional J dependent terms in the renormalised action. In this case the correlation functions, given by

$$\frac{1}{Z[0]}\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z[J] \tag{2.5}$$

equal to $<\phi(x)\phi(y)>$ at the bare level will be completely unaltered by the renormalization procedure.

2.2 The Polchinski flow equation

The Polchinski flow equation [17] governs the flow of S_Λ under varying Λ. Here we present a constructive proof following the methods found in ref. [18]. The partition function for N component scalar field theory with propagator Δ^{-1}, arbitrary bare action $S_{\Lambda_o}[\phi]$ and source J, can be written

$$Z[J] = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi_a.\Delta^{-1}.\phi_a - S_{\Lambda_o} + J_a.\phi_a} \tag{2.6}$$

where the field ϕ has a flavour index a which runs from 1 to N. Consider the following integral which we will show reduces to (2.6). These fields ϕ_{IR} and ϕ_{UV} are integrated over freely.

$$Z[J] = \int \mathcal{D}\phi_{IR}\mathcal{D}\phi_{UV} \; e^{-\frac{1}{2}\phi_{UV}.\Delta_{UV}^{-1}.\phi_{UV} - \frac{1}{2}\phi_{IR}.\Delta_{IR}^{-1}.\phi_{IR} - S_{\Lambda_o}[\phi_{IR}+\phi_{UV}] + J.(\phi_{IR}+\phi_{UV})} \tag{2.7}$$

where

$$\Delta(p) = \Delta_{IR}(p) + \Delta_{UV}(p) \tag{2.8}$$

and

$$\phi_a = \phi_{IR} + \phi_{UV}. \tag{2.9}$$

For simplicity of notation we have suppressed the flavour index a on the right side of (2.9). Physically, we interpret the ϕ_{IR} fields as the momentum modes higher than Λ[2] and the ϕ_{UV} fields as the modes that are lower than Λ. By changing variables in (2.7), $\phi_{IR} = \phi - \phi_{UV}$ and $\phi_{UV} = \phi'_{UV} + \Delta_{UV}\Delta^{-1}.\phi$, and performing a Gaussian integral we reproduce the partition function in (2.6) up to a constant of proportionality.

Now consider integrating only over the higher modes:

$$Z_\Lambda[J, \phi_{UV}] = \int \mathcal{D}\phi_{IR} \; e^{-\frac{1}{2}\phi_{IR}.\Delta_{IR}^{-1}.\phi_{IR} - S_{\Lambda_o}[\phi_{IR}+\phi_{UV}] + J.(\phi_{IR}+\phi_{UV})}. \tag{2.10}$$

Using (2.9) to eliminate ϕ_{IR} in favour of ϕ, it is straightforward to show that Z_Λ does not depend on both J and ϕ_{UV} independently in the sense that

$$Z_\Lambda[J, \phi_{UV}] = e^{\frac{1}{2}J.\Delta_{IR}.J + J.\phi_{UV} - S_\Lambda[\Delta_{IR}.J + \phi_{UV}]} \tag{2.11}$$

[2] i.e. $q^2 > \Lambda^2$,

for some functional S_Λ which depends only on the sum $\Delta_{IR}.J + \phi_{UV}$. If we restrict the support of J to low energy modes only, *i.e.* we set $\Delta_{IR}.J = 0$, (2.11) simplifies. However from (2.10) and (2.7),

$$Z[J] = \int \mathcal{D}\phi_{UV}\ Z_\Lambda[J, \phi_{UV}] e^{-\frac{1}{2}\phi_{UV}.\Delta^{-1}_{UV}.\phi_{UV}}, \tag{2.12}$$

and we see that S_Λ is nothing more than the interaction part of the Wilsonian effective action, as defined in (2.3). The exact flow equation follows readily from the fact that (2.10) depends on Λ only through the $\phi_{IR}.\Delta^{-1}_{IR}.\phi_{IR}$ term. Thus, differentiating Z_Λ with respect to Λ we immediately obtain the flow equation for Z_Λ:

$$\frac{\partial}{\partial\Lambda} Z_\Lambda[\phi_{UV}, J] = -\frac{1}{2}\left(\frac{\delta}{\delta J} - \phi_{UV}\right) \cdot \left(\frac{\partial}{\partial\Lambda}\Delta^{-1}_{IR}\right) \cdot \left(\frac{\delta}{\delta J} - \phi_{UV}\right) Z_\Lambda. \tag{2.13}$$

Substituting (2.11), performing a change of variables $\Phi = \Delta_{IR}.J + \phi_{UV}$ and relabelling $\phi = \Phi$, yields Polchinski's version of the Wilson flow equation [17]:

$$\frac{\partial S_\Lambda}{\partial\Lambda} = \frac{1}{2}\frac{\delta S_\Lambda}{\delta\phi}.\frac{\partial\Delta_{UV}}{\partial\Lambda}.\frac{\delta S_\Lambda}{\delta\phi} - \frac{1}{2}\mathrm{tr}\left(\frac{\partial\Delta_{UV}}{\partial\Lambda}.\frac{\delta^2 S_\Lambda}{\delta\phi\delta\phi}\right) \tag{2.14}$$

where the trace represents an integral over position space. In terms of the cutoff function, $C = C_{UV}$, this is written

$$\frac{\partial S_\Lambda}{\partial\Lambda} = \frac{1}{\Lambda^3}\left(\mathrm{tr}\left(C'.\frac{\delta^2 S_\Lambda}{\delta\phi_a\delta\phi_a}\right) - \frac{\delta S_\Lambda}{\delta\phi_a}.C'.\frac{\delta S_\Lambda}{\delta\phi_a}\right). \tag{2.15}$$

where we have reintroduced the flavour index.

2.3 The Legendre flow equation

The Legendre flow equation will prove particularly useful for us as it can be solved exactly in the large N limit. However, we will extensively use the Legendre flow equation for Fermions and thus omit a calculation of the large N limit for scalar field theory[3]. Consider the action with kinetic term modified to include an infra-red (IR) cutoff, as described in section 2.1. Then

[3] we will however demonstrate that the LPA provides exact exponents in this limit

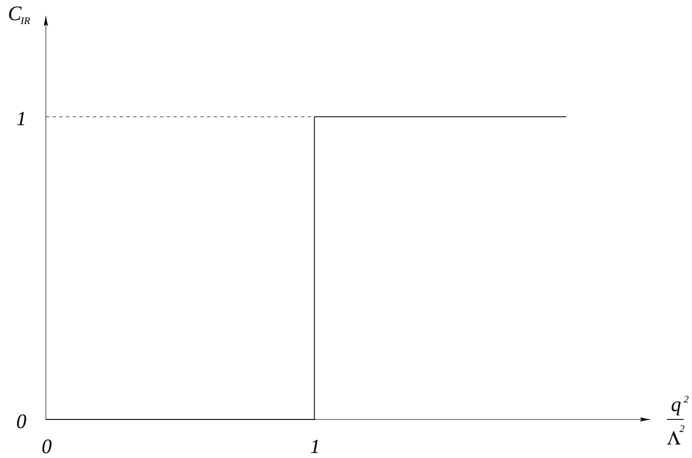

Figure 2.3: An infra-red cutoff function

the generating functional can be written

$$Z[J] = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi_a.\Delta_{IR}^{-1}.\phi_a - S_\Lambda + J_a.\phi_a} \tag{2.16}$$

where a runs from 1 to N as before, but here $\Delta_{IR} = \frac{C_{IR}}{q^2}$ where C_{IR} might take the form shown in figure 2.3. We begin by defining the generating functional of connected Greens functions by $W[J] = \ln(Z[J])$ and introducing the Legendre effective action, $\Gamma_\Lambda[\phi^c]$,

$$\Gamma_\Lambda[\phi^c] + \frac{1}{2}\phi^c_a.\Delta_{IR}^{-1}.\phi^c_a = -W[J] + J_a.\phi^c_a, \tag{2.17}$$

where in the classical limit $h \to 0$, $\Gamma_\Lambda \to S_\Lambda$. The classical field is defined as

$$\phi^c_a = \frac{\delta W}{\delta J_a}, \tag{2.18}$$

and corresponds to the expectation value of the field. Differentiating (2.17) twice with respect to ϕ^c, and then separately twice with respect to J, we derive

$$\frac{\delta^2 W}{\delta J_a \delta J_b} = \left(\frac{\delta^2 \Gamma_\Lambda}{\delta\phi^c_a \delta\phi^c_b} + \delta_{ab}\Delta_{IR}^{-1} \right)^{-1}. \tag{2.19}$$

We also differentiate (2.17) once with respect to Λ, to find

$$\left.\frac{\partial \Gamma_\Lambda}{\partial \Lambda}\right|_{\phi^c} + \frac{1}{2}\phi^c_a.\frac{\partial \Delta_{IR}^{-1}}{\partial \Lambda}.\phi^c_a = -\left.\frac{\partial W}{\partial \Lambda}\right|_J. \tag{2.20}$$

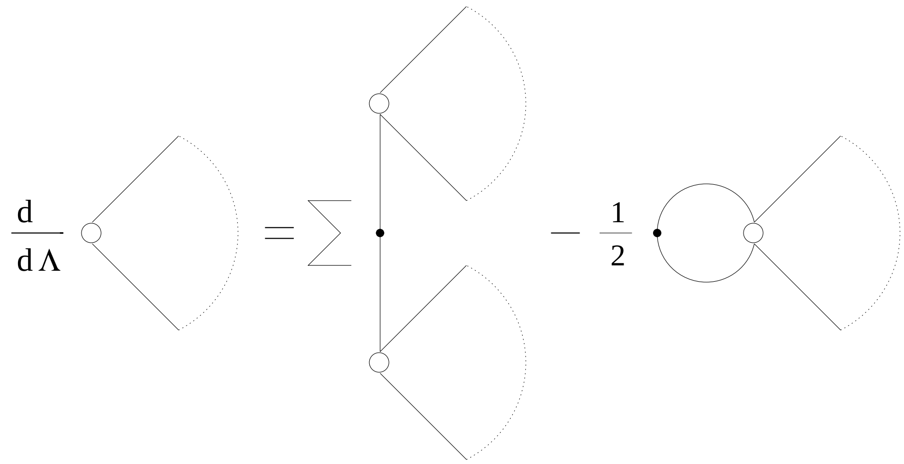

Figure 2.4: The flow equations for the vertices of the generating functional W_Λ

Finally we find an expression for the right side of (2.20) by differentiating the partition function with respect to Λ,

$$\frac{\partial W}{\partial \Lambda} = -\frac{1}{2}\frac{\delta W}{\delta J}.\frac{\partial \Delta_{IR}^{-1}}{\partial \Lambda}.\frac{\delta W}{\delta J} - \frac{1}{2}\mathrm{tr}\left(\frac{\partial \Delta_{IR}^{-1}}{\partial \Lambda}.\frac{\delta^2 W}{\delta J \delta J}\right) \tag{2.21}$$

which then leads to the exact Legendre flow equation upon substituting (2.19),

$$\frac{\partial \Gamma_\Lambda}{\partial \Lambda} = \frac{1}{2}\mathrm{tr}\left(\frac{\partial \Delta_{IR}^{-1}}{\partial \Lambda}.(A^{-1})_{aa}\right) \tag{2.22}$$

where

$$A_{ab} = \frac{\delta^2 \Gamma_\Lambda}{\delta \phi_a^c \delta \phi_b^c} + \delta_{ab}\Delta_{IR}^{-1}. \tag{2.23}$$

The equations are best appreciated graphically. We view (2.21) as depicted in figure 2.4, which shows how the n-point functions $W_n(p_1, \cdots, p_n)$ of W_Λ evolve. The vertices are drawn as open circles, whereas the black dots represent the two point function, $\frac{\partial \Delta_{IR}^{-1}}{\partial \Lambda}$. In the case of a sharp cutoff these represent a delta function restricting momenta to $q = \Lambda$. The Polchinski flow equation (2.14) can be viewed in a similar manner. In fact the Polchinski and Legendre flow equations are intimately linked [18]. In particular the Wilsonian action behaves like the generator of the connected diagrams and these effective actions are thus found to be related by a generalised Legendre transform,

$$S_\Lambda[\phi_a] = \Gamma_\Lambda[\phi_a^c] + \frac{1}{2}(\phi^c - \phi)_a.\Delta_{IR}^{-1}.(\phi^c - \phi)_a. \tag{2.24}$$

Eliminating Γ_Λ from (2.22) above, we retrieve the Polchinski flow equation (2.14), if we impose (2.4). A more detailed discussion of this relationship can be found in the literature [18].

2.4 Application of the Local Potential Approximation

The Local Potential Approximation (LPA) can be applied with considerable success to either the Polchinski or Legendre flow equation. Here we choose to focus our attention on the Polchinski flow equation [17]. Consider the flow equation in terms of the cutoff function, C, given by (2.15). Here we impose the LPA by writing S_{ren} as (1.45), thus

$$\frac{\delta S_\Lambda}{\delta \phi_a}.C'.\frac{\delta S_\Lambda}{\delta \phi_a} = \int d^D x d^D y \; V'(\phi(x)) C'_{xy} V'(\phi(y)). \tag{2.25}$$

Fourier transforming,

$$\begin{aligned} C'_{xy} &= \frac{1}{(2\pi)^D} \int d^D q \; e^{iq(x-y)} C'\left(\frac{q^2}{\Lambda^2}\right) & (2.26) \\ &= C'\left(-\frac{\partial^2}{\Lambda^2}\right) \delta(x-y). & (2.27) \end{aligned}$$

Considering $C'(-\frac{\partial^2}{\Lambda^2})$ as an expansion in powers of $(-\frac{\partial^2}{\Lambda^2})$ we discard all but the leading term, $C'(0)$. Then we find that (2.25) can be written

$$\frac{\delta S_\Lambda}{\delta \phi_a}.C'.\frac{\delta S_\Lambda}{\delta \phi_a} = C'(0) \int d^D x \; V'^2(\phi(x)). \tag{2.28}$$

Later we will assume that $C'(0) < 0$. In principle the $C'(0)$ could well be taken to vanish however we take $C'(0) < 0$ for the approximation to proceed. This may be considered physically unreasonable however the flow equation is ultimately independent of $C'(0)$, suggesting that this assumption may actually be sensible. Similarly, we simplify the first term in (2.15) using (2.26) to find

$$\begin{aligned} \mathrm{tr}\left(C'.\frac{\delta^2 S_\Lambda}{\delta \phi_a(x) \delta \phi_a(y)}\right) &= \int d^D x d^D y \; C'_{xy} \frac{\delta^2 S_\Lambda}{\delta \phi_a(x) \delta \phi_a(y)} & (2.29) \\ &= A \Lambda^D \int d^D x \; V''(\phi(x)) & (2.30) \end{aligned}$$

where we define the dimensionless A through

$$A\Lambda^D = \frac{1}{(2\pi)^D}\int d^Dq\; C'\left(\frac{q^2}{\Lambda^2}\right) < 0. \tag{2.31}$$

Then the 'reduced Polchinski equation' reads

$$\frac{\partial V}{\partial \Lambda} = A\Lambda^{D-3}V'' - \frac{C'(0)}{\Lambda^3}V'^2. \tag{2.32}$$

Scaling to dimensionless quantities via $\phi \to \phi\Lambda^{\frac{1}{2}(D-2)}\sqrt{|A|}$ (where as mentioned in section 1.12 $\eta = 0$ in the LPA) and $V \to V\Lambda^D\frac{|A|}{|C'(0)|}$ and using renormalization time as defined by (1.29) we finally arrive at a cutoff independent, and hence scheme independent, flow equation, which will be the subject of our attention:

$$\frac{\partial V}{\partial t} = DV - \frac{1}{2}(D-2)\phi_a\frac{\partial V}{\partial \phi_a} - \left(\frac{\partial V}{\partial \phi_a}\right)^2 + \frac{\partial^2 V}{\partial \phi_a^2}. \tag{2.33}$$

The physical predictions (critical exponents) are independent of the scaling[4] of ϕ and V, although the form of the potential will be altered.

Given the definition of a fixed point as $V = V_*$ such that $\frac{\partial V_*}{\partial t} = 0$, it is immediately noticed that there are two trivial solutions, the Gaussian and High Temperature fixed points which are given by

$$V_*^G = 0 \tag{2.34}$$

and

$$V_*^{HT} = \frac{1}{2}\phi^2 - \frac{N}{D} \tag{2.35}$$

respectively. We will shortly justify the name High Temperature for the latter. In addition we will find at least one non-trivial fixed point below four dimensions. From (2.33) we can deduce that with the exception of the Gaussian solution, in the large ϕ regime, $V_*(\phi) \sim \frac{1}{2}\phi^2$.

To compute the critical exponents we perturb about the fixed point solution as described in chapter one. We substitute $V(\phi,t) = V_*(\phi) + \epsilon\delta V(\phi,t)$ into (2.33) and work to $O(\epsilon)$. Then by separation of variables we find that the potential takes the form of (1.31) as claimed and that

[4] *i.e.* under $\phi \to \mathcal{A}\phi$ and $V \to \mathcal{B}V$ for arbitrary constants $\mathcal{A}$ and $\mathcal{B}$

the $f_n(\phi)$ satisfy

$$\frac{\partial^2 f_n}{\partial \phi_a^2} = (\lambda_n - D) f_n + \frac{1}{2}(D-2)\phi_a \frac{\partial f_n}{\partial \phi_a} + 2\frac{\partial V_*}{\partial \phi_a}\frac{\partial f_n}{\partial \phi_a}. \tag{2.36}$$

For the trivial fixed points we can easily solve to extract the λ_n. Restricting to $N = 1$, we observe that the solutions take the form of a polynomial, $f_n(\phi) = \phi^A + B(A)\phi^{A-2} + C(A)\phi^{A-4} + \cdots$, with $A = 2n$ for integer n such that $f_n(\phi)$ has no divergences. The λ_n can then be written in terms of A. In particular, for the trivial fixed points we find

$$\lambda_n^G = D - n(D-2) \tag{2.37}$$

and

$$\lambda_n^{HT} = D - n(D+2). \tag{2.38}$$

Note that the eigenvalues for the Gaussian fixed point are the classical dimension of the couplings. We observe, that for the Gaussian fixed point in any dimension, the leading exponent defined by (1.34) is[5] $\nu = \frac{1}{2}$. However, for the High Temperature fixed point and for all $D > 0$ we find that all the $\lambda_n < 0$ implying that there are just irrelevant directions around this fixed point. In addition, the exponents of the High Temperature fixed point are independent of N corresponding to an infinitely massive theory with no propagation.

2.5 Flow and β-functions

For illustration, we consider the simple case of the Gaussian fixed point perturbed by the mass operator, for a single scalar field. Thus, using the notation introduced in section 1.10, we set (for $N = 1$) $V(\phi, t) = g^1(t) + \frac{1}{2}g^2(t)\phi^2$. The β-functions for g^1 and g^2 then follow easily by substitution in (2.33):

$$\beta^1(g) = \frac{\partial g^1}{\partial t} = Dg^1 + g^2 \tag{2.39}$$

and

$$\beta^2(g) = \frac{\partial g^2}{\partial t} = 2g^2(1 - g^2). \tag{2.40}$$

[5]this result is in direct agreement with mean field theory [5]

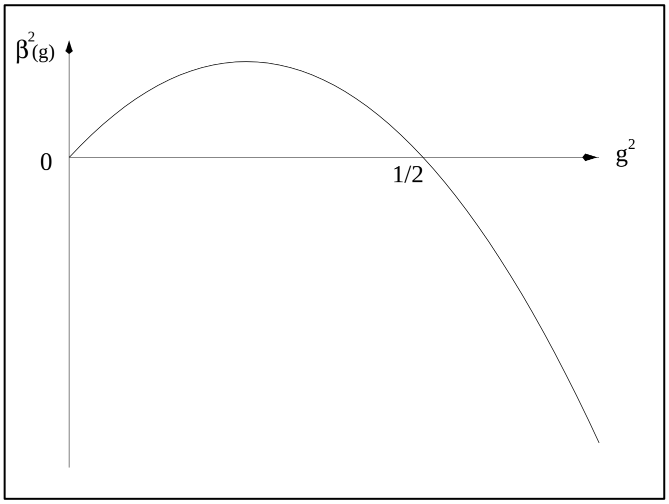

Figure 2.5: The β-function for the mass term of three dimensional one component Z_2-invariant scalar field theory

These can be solved for $g^1(t)$ and $g^2(t)$. As expected, at the Gaussian ($g^1 = g^2 = 0$) and the High Temperature ($g^1 = -\frac{1}{D}$ and $g^2 = 1$) fixed points, the β-functions vanish. Now we can plot the β-function for the mass (figure 2.5). We deduce that if we add a small mass term to the Gaussian we will flow into the High Temperature fixed point. In other words the mass term corresponds to a relevant perturbation for the Gaussian fixed point, but an irrelevant perturbation for the High Temperature fixed point.

2.6 Truncations

We are interested in finding non-trivial fixed points. Here, we outline the method of truncations for one component theory. In this approximation method (in addition to the LPA) we expand $V_*(\phi)$ in powers of ϕ and truncate the series at some finite power, $2p$ ($p = \frac{1}{2}, 1, \frac{3}{2}, \cdots$). Begin by considering V_* as an infinite series in powers of ϕ

$$V_* = \sum_{n=0}^{\infty} V_n \phi^n \tag{2.41}$$

such that the fixed point version of (2.33) yields

$$\frac{n}{2}(D-2)V_n - DV_n = (n+1)(n+2)V_{n+2} - \sum_{u+v=n+2} uvV_uV_v \tag{2.42}$$

as the coefficients of a general power, ϕ^n. In the last term $u \geq 1$ and $v \geq 1$ are integers. We will consider a Z_2-invariant theory (even $V(\phi)$) which requires $V_n = 0$ for odd n.

Similarly we write an eigenperturbation as a series[6],

$$f = \sum_{n=0}^{\infty} f_n \phi^n \tag{2.43}$$

such that the coefficients of ϕ^n in (2.36) yields

$$\frac{n}{2}(D-2)f_n - (D-\lambda)f_n = (n+1)(n+2)f_{n+2} - 2\sum_{u+v=n+2} uvV_uf_v. \tag{2.44}$$

Given that we truncate the series at a power $2p$ we find that the eigenvalues are found by solving the determinant of a $(p+1)\times(p+1)$ matrix,

$$\det \begin{pmatrix} D-\lambda & 2 & 0 & \cdots \\ 0 & 2-8V_2-\lambda & 12 & \cdots \\ 0 & -16V_4 & 4-D-16V_2-\lambda & \cdots \\ 0 & -24V_6 & -32V_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0. \tag{2.45}$$

For illustration consider the simplest case of $p = 2$, which leads to three simultaneous equations for the V_i:

$$V_0 = -\frac{2}{D}V_2, \tag{2.46}$$

$$V_2 = 2V_2^2 - 6V_4, \tag{2.47}$$

and

$$(4-D)V_4 = 16V_2V_4. \tag{2.48}$$

Solving, we reproduce both the Gaussian and High Temperature fixed points. Additionally we

[6]the f_n defined here are distinct to the eigenfunctions used in the latter part of section 2.4

find a non-trivial fixed point,

$$V_*^{NT} = \frac{D-4}{8D} + \frac{4-D}{16}\phi^2 + \frac{D^2-16}{768}\phi^4. \tag{2.49}$$

Immediately we observe that it becomes degenerate with the Gaussian in 4 dimensions and the High Temperature in -4 dimensions. The former is observed generally. The first three eigenvalues for the Gaussian and High Temperature fixed points, given by (2.37) and (2.38), are reproduced exactly. In principle the non-trivial fixed point in three dimensions would have $\lambda = 3, \frac{3}{4} \pm \frac{1}{2}\sqrt{\frac{37}{4}}$, however (2.49) is actually unbounded from below. This is unphysical and will be seen to be an artefact of the truncation.

Generally, the method of truncations proves problematic. The exponents calculated in this case are found to initially converge but then fail to converge further with increasing p [19]. Most notably spurious solutions are generated for $p > 2$, and some procedure must be sought to distinguish the true fixed points. In some cases it has been found that some improvement can be made by expanding around the minimum of the potential [20]. The truncation method has been found to yield imaginary exponents (ω here) which would imply exotic flows near the fixed point. However, later we will find that the exponents are expected to be real, excluding the possibility of exotic flows.

2.7 Shooting

Exact non-trivial fixed points (within the LPA) are found by solving (2.33) numerically. We use a method known as shooting [13]. We see that there is a one parameter (denoted s) set of solutions, corresponding to a second order differential equation with a single boundary condition. For a given value of s we integrate out for $V(\phi)$ until it diverges, generally at finite $\phi = \phi_c$. Then we 'scan' over s and look for non-divergent solutions at $s = s_*$, which correspond to our fixed points[7]. For consistency, we consider again a Z_2-invariant one component theory

[7] this is a requirement of Griffiths analyticity combined with the required analyticity at the origin

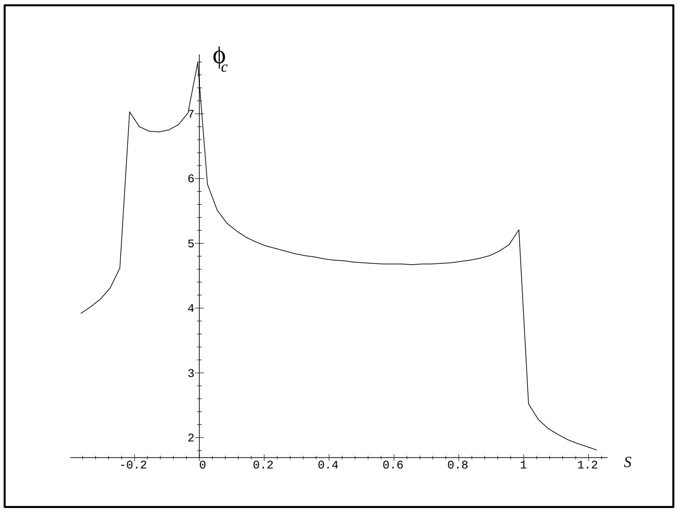

Figure 2.6: Shooting: ϕ_c v s

such that V is even in ϕ and then set our boundary conditions as

$$V(0) = -\frac{s}{D} \tag{2.50}$$

and

$$V'(0) = 0 \tag{2.51}$$

where the $s = V''(0)$ and prime denotes differentiation with respect to ϕ. The first condition is a translation directly from (2.33) and the second is due to the symmetry of the theory. A plot of ϕ_c against s is shown for three dimensions, in figure 2.6. The fixed points occur for diverging ϕ_c; as expected from (2.34) and (2.35) the Gaussian occurs at $s_*^G = 0$ and the High Temperature occurs at $s_*^{HT} = 1$. The non-trivial Wilson-Fisher fixed point [16] is observed at $s_*^{WF} = -0.229$. In figure 2.7, we plot V_*^{WF}.

Varying the dimension, we observe that there are no non-trivial fixed points at or above four dimensions. In exactly four dimensions we find the Wilson-Fisher fixed point is degenerate with the Gaussian, where the Gaussian has a marginal eigenvalue, $\lambda_2 = 0$. With decreasing dimension, the Wilson-Fisher fixed point occurs for decreasing s, such that it moves away from the Gaussian (left in figure 2.6), and new non-trivial fixed points are created which are initially degenerate with the Gaussian for dimensions where the $\lambda_n^G = 0$ $(n = 2, 3, \cdots)$. These marginal

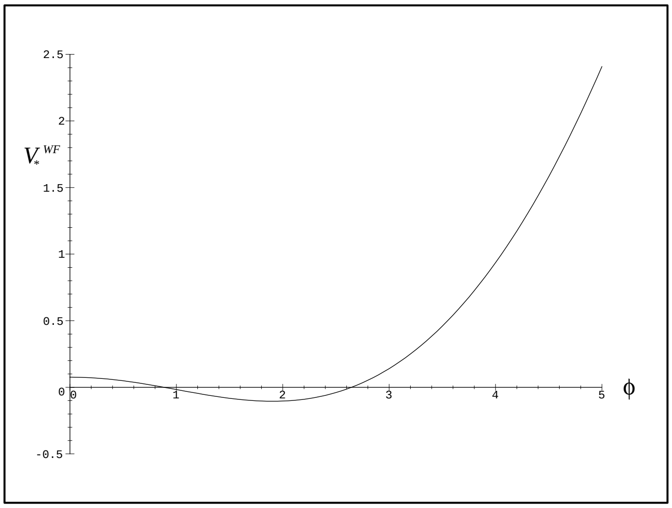

Figure 2.7: The Wilson-Fisher fixed point

eigenvalues allow the addition of the corresponding perturbation to the Gaussian, creating a new fixed point as previously discussed. The dimensions are given by

$$D = \frac{2(n+1)}{n}. \tag{2.52}$$

Hence, below four dimensions we see at least one non-trivial fixed point whereas below three dimensions we see at least two (Appendix 2A). At and below two dimensions we observe a continuous line of 'oscillating' fixed points. The behaviour of s_* with dimension D is illustrated for the Wilson-Fisher and two subleading fixed points in figure 2.8.

Finally (2.36) is solved for the exponents in a similar manner. Here we impose two conditions,

$$f(0) = 1 \tag{2.53}$$

and

$$f'(0) = 0 \tag{2.54}$$

where the first is an arbitrary normalisation and the second is an imposition of our choosing (we are restricting to even perturbations). The solutions generally diverge and are found to depend on the eigenvalue, λ. Thus, similar to before we sweep through a range of λ and look for acceptable (*i.e.* non-divergent) $f(\phi)$. For the Wilson-Fisher fixed point, the largest values

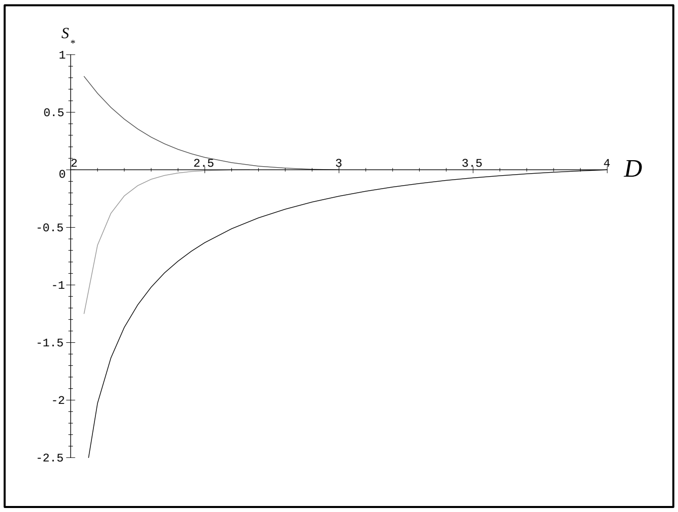

Figure 2.8: A plot of s_* v D for the Wilson-Fisher and two subleading fixed points

yield $\nu = 0.6496$ and $\omega = 0.6557$. Later, these will be compared to other leading methods.

2.8 Variation

We begin by introducing $\rho(\phi, t) = e^{-V(\phi,t)}$ and $G(\phi) = e^{-\frac{1}{4}(D-2)\phi_a^2}$ such that the Polchinski equation (2.33) can be written [21,22]

$$a^N G \frac{\partial \rho}{\partial t} = -\frac{\delta \mathcal{F}}{\delta \rho} \tag{2.55}$$

where

$$\mathcal{F}[\rho] = a^N \int d^N\phi \; G \left(\frac{1}{2} \left(\frac{\partial \rho}{\partial \phi_a} \right)^2 + \frac{D}{4} \rho^2 \left(1 - 2 \ln \rho\right) \right) \tag{2.56}$$

and $a > 0$ is a normalisation factor for the ϕ measure to be determined later. Then the fixed point is defined by $\frac{\delta \mathcal{F}}{\delta \rho} = 0$. An interesting reparameterisation of (2.56) is discussed in Appendix 2B.

As discussed in chapter one we let $g^i(t)$ be a complete set of parameters (the coupling constants) for V. Away from fixed points these will be infinite in number. Then using (2.55), we write

$$\frac{\partial \mathcal{F}}{\partial g^i} = \int d^N\phi \; \frac{\delta \mathcal{F}}{\delta \rho}\frac{\partial \rho}{\partial g^i} \tag{2.57}$$

$$= -a^N \int d^N\phi \; G \frac{\partial \rho}{\partial t}\frac{\partial \rho}{\partial g^i} \tag{2.58}$$

$$= -M_{ij}\beta^j \tag{2.59}$$

where β^j are the β-functions given by (1.41) and M_{ij} is defined by

$$M_{ij} = a^N \int d^N\phi \; G \frac{\partial \rho}{\partial g^i}\frac{\partial \rho}{\partial g^j}. \tag{2.60}$$

Perturbing about the fixed point solution by writing $g^j = g^j_* + \delta g^j = g^j_* + b^j e^{\lambda t}$, (2.59) yields

$$\frac{\partial^2 \mathcal{F}}{\partial g^i \partial g^j}\delta g^j = \frac{\partial \mathcal{F}}{\partial g^i} \tag{2.61}$$

$$= -\lambda M_{ij} b^j e^{\lambda t} \tag{2.62}$$

such that (assuming that b^j is not a null vector) we calculate the eigenvalues, λ, by solving

$$\det\left(\lambda M_{ij} + \frac{\partial^2 \mathcal{F}}{\partial g^i \partial g^j}\right) = 0. \tag{2.63}$$

The fixed point equation, $\frac{\delta \mathcal{F}}{\delta \rho} = 0$, suggests the possibility of approximating ρ by a variational ansatz, $\rho = f(\phi, g^i)$ where i runs from 1 to a finite M and f is of our choosing. Then interpreting the functional derivative in (2.55) to include only variations in this restricted set, we arrive again at (2.59) where however, indices only run from 1 to M. Thus geometrically, we restrict the flows to a submanifold $\mathcal{M}$ parametrised by $g^1, \cdots, g^M$.

It is important that these methods reproduce at least the trivial fixed points accurately. Here we will rederive the Gaussian and High Temperature fixed points with the leading exponent using the methods described above. We take $V = g^1(t) + \frac{1}{2}g^2(t)\phi^2$ as an ansatz and minimise the functional in (2.56)

$$\frac{\partial \mathcal{F}}{\partial g^i} = a^N \int d^N\phi \; G\left(\rho' \frac{\partial \rho'}{\partial g^i} - D\rho \ln \rho \frac{\partial \rho}{\partial g^i}\right) \tag{2.64}$$

$$= 0 \tag{2.65}$$

to arrive at a pair of simultaneous equations for g^1 and g^2. Performing the resulting Gaussian integrals (Appendix 2C), we arrive at the following conditions

$$g^2(D+2g^2)+2g^1D[\frac{1}{2}(D-2)+2g^2]=0 \tag{2.66}$$

and

$$3g^2(D+2g^2)+2(g^1D-2g^2)[\frac{1}{2}(D-2)+2g^2]=0. \tag{2.67}$$

Solving these leads directly to the Gaussian ($g^1=g^2=0$) and High Temperature ($g^1=-\frac{1}{D}$ and $g^2=1$) solutions. Then differentiating (2.64) with respect to g^i and substituting (2.60) into (2.63) and again performing the resultant Gaussian integrals we reproduce the leading exponents for the trivial fixed points. For the Gaussian (2.63) leads to $(\lambda-D)(\lambda-2)=0$ and for the High Temperature we find $(\lambda-D)(\lambda+2)=0$.

2.9 The Wilson-Fisher fixed point by Variation

The true test of the variation method lies in the accuracy with which it can reproduce the Wilson-Fisher fixed point [22]. Thus, working again with a Z_2-invariant one component theory and using the simplest non-trivial ansatz, $V(\phi,t)=g^1(t)+\frac{1}{2}g^2(t)\phi^2+g^3(t)\phi^4$, we find an approximate solution to the Wilson-Fisher fixed point, corresponding to $g^1_*=0.05479$, $g^2_*=-0.13488$ and $g^3_*=0.00773$. In figure 2.9, we plot the resulting form of ρ_* and compare it to the exact non-trivial fixed point solution[8], solved by shooting. Solving (2.63), we obtain, apart from $\lambda=3$, $\nu=0.6347$ from the positive eigenvalue, and $\omega=0.6093$ from the remaining eigenvalue. These are 2% and 8% away from the exact[8] values respectively.

Clearly, it may be possible to find 'spurious fixed points' that do not well approximate the exact solutions, but these will not be stable under changing or improving the ansatz manifold $\mathcal{M}$. In fact as $\mathcal{M}$ improves, such fixed points (if indeed there are any) will disappear. These characteristics are in marked contrast to the general situation for truncations, as discussed earlier. To improve $\mathcal{M}$ we may simply extend the series to include a ϕ^6 term, or alternatively appeal to (2.33) to build a more sophisticated ansatz. In practice such parameterisations are

[8] exact within the LPA

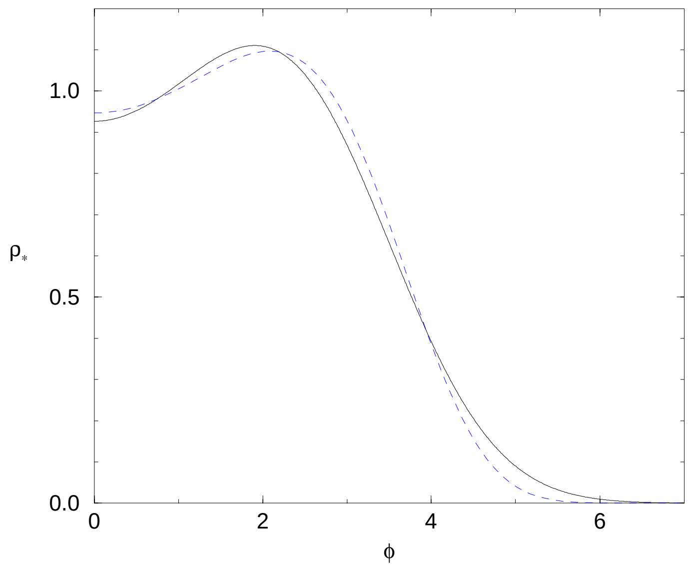

Figure 2.9: The simplest polynomial variational approximation to the Wilson-Fisher fixed point (dashed line) compared to the exact solution (full line)

increasingly difficult to solve and the results presented here are of sufficient accuracy. As applied here, the variational method provides no real advantage over shooting. The true potential of this method lies in the relative ease in which approximations for global flows may be solved and approximate solutions found for more than one invariant.

2.10 The large N limit of the LPA

The large N limit of the LPA to the Polchinski equation can be solved analytically and compared with known exact results. It is thus a good testing ground for the LPA. In fact, we will find that the LPA yields exact results in this limit giving us added confidence in the reliability of this approximation. Using other methods [1] various authors have found that, for large N, there exists a fixed point in dimensions $2 < D < 4$ with $\nu = \frac{1}{D-2}$. Here we reproduce these results, using the methods of Morris *et. al.* [23].

We begin by defining $z(x) = \phi_a^2(x) > 0$ such that within the LPA V can be written as a function of z and t only, corresponding to an $O(N)$ invariant theory. Then the Polchinski equation (2.33)

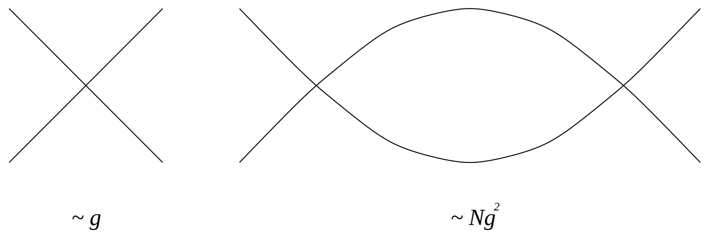

Figure 2.10: N-dependence of the two leading contributions to the four point function

can be written

$$\frac{\partial V}{\partial t} = DV - ((D-2)z + 2N)\frac{\partial V}{\partial z} + 4z\frac{\partial^2 V}{\partial z^2} - 4z\left(\frac{\partial V}{\partial z}\right)^2. \tag{2.68}$$

We observe $V = 0$ and $V = \frac{1}{2}z - \frac{N}{D}$ as the trivial fixed points, however concentrate on non-trivial solutions. By considering figure 2.10 (due to $g\phi^4$) and requiring a non-trivial finite limit, $g \sim Ng^2$ which implies that $g \sim \frac{1}{N}$. Then $m\phi^2 \sim \frac{1}{N}\phi^4$ which implies $\phi^2 \sim N$ and thus $V \sim N$ where the mass is N independent. Thus changing variables so as to scale out the N dependences, and taking the infinite N limit the second order derivative in (2.68) vanishes. Then, differentiating with respect to z and defining $W = \frac{\partial V}{\partial z}$ we arrive at the following flow equation,

$$\frac{\partial W}{\partial t} + ((D-2)z + 8zW - 2)\frac{\partial W}{\partial z} = 2W - 4W^2. \tag{2.69}$$

Alternatively, shifting to a stationary point in V, z_o, (thus excluding the High Temperature solution) and defining $z = z_o + y$, we arrive at [23]

$$\dot{W} - \dot{z}_o W' + (D-2)(z_o + y)W' - 2W = 2W' - 8(z_o + y)WW' - 4W^2. \tag{2.70}$$

Here the dot denotes differentiation with respect to t and the prime denotes differentiation with respect to y.

However, because we have shifted to a minimum we require $W(0,t) = 0$ in (2.70), which yields $W'(0,t) = 0$ and thus $W(y,t) = 0$ (by expanding in powers of y), unless

$$\dot{z}_o = (D-2)z_o - 2. \tag{2.71}$$

Substituting this non-Gaussian solution back into (2.70) yields

$$\dot{W} + (D-2)yW' - 2W = -8(z_o + y)WW' - 4W^2. \tag{2.72}$$

Then, expanding in powers of y,

$$W(y,t) = \sum_{n=1}^{\infty} W_n(t)y^n, \tag{2.73}$$

yields a series of equations,

$$\dot{W}_1 + ((D-2)-2)W_1 = -8z_oW_1^2, \tag{2.74}$$

$$\dot{W}_2 + (2(D-2)-2)W_2 = -24z_oW_1W_2 - 8W_1^2, \tag{2.75}$$

$$\dot{W}_n + (n(D-2)-2)W_n = -8(n+1)z_oW_1W_n \cdots. \tag{2.76}$$

The final equation holds for $n > 1$ and the dots stand for terms containing products of W_m's with $m < n$. Thus the $W_n(t)$ are all soluble in terms of W_m, with $m < n$, and $z_o(t)$, and truncations to some finite n are exact.

Now consider a fixed point, $W(y,t) = W(y)$ and $z_o(t) = z_o$. Then by (2.71)

$$z_o = \frac{2}{D-2}, \tag{2.77}$$

however as z_o is the original field squared it must be positive, requiring $D > 2$. Note that since there is only one solution for z_o, $W(y)$ can only cross the axis once. Substituting this into (2.74) yields $W_1 = 0$ (the Gaussian) and

$$W_1 = \frac{(D-2)(4-D)}{16}. \tag{2.78}$$

Since $W(y)$ crosses the y-axis only once, we must have $W_1 > 0$ which implies we must have $D < 4$ (as we have established $D > 2$), so that $W(y) > 0$ for $y > 0$. Otherwise the potential is unbounded below. Substituting (2.78) into the series of equations (2.74) to (2.76) we see that the W_n exist and are unique (*e.g.* $W_2 = \frac{6}{8-5D}W_1^2$).

We deduce the leading exponent by perturbing the z_o in (2.71) about the fixed point solution,

$z_o = z_{o*} + \delta z_o$, yielding $\delta \dot{z}_o = (D-2)\delta z_o$, such that there is an eigenperturbation $\delta z_o \propto e^{\lambda_0 t}$, with $\lambda_0 = D - 2$ implying

$$\nu = \frac{1}{D-2}, \qquad 2 < D < 4. \tag{2.79}$$

Now we can systematically generate the subleading exponents by perturbing about fixed point solutions in (2.74) to (2.76). For example, by writing $W_1 = W_{1*} + \delta W_1$ in (2.74) (with $\delta z_o = 0$), we arrive at $\delta \dot{W}_1 = (D-4)\delta W_1$, or $\delta W_1 \propto e^{(D-4)t}$, yielding $\lambda_1 = D-4$. Then in (2.75) we find a new perturbation which takes the form $\delta z_o(t) = 0$, $\delta W_1(t) = 0$ and $\delta W_2(t) \propto e^{(D-6)t}$, yielding $\lambda_2 = D-6$. In this way we generate the set of eigenvalues in the large N limit, $\lambda_n = D - 2n$ for $n = 1, 2, \cdots$.

2.11 Discussion

Referring to the numerical methods, the most outstanding feature is that they allow us to exhaustively search the entire infinite dimensional space using a single parameter and to make physical predictions for complex systems which can be measured experimentally. The results for ν and ω found by shooting and the variational method compare favourably with other leading methods (table 2.1) [1]. It is not yet known how to apply the momentum expansion (beyond LPA) to the Polchinski flow equation in a reparameterisation invariant [15] way. However the momentum expansion at $O(p^2)$ has been applied to the Legendre flow equation and shows a modest improvement in the exponents, over the LPA [14]. Traditionally it is difficult to estimate the errors introduced using these approximation schemes, other than by comparison with the other methods, though recently progress has been made [14]. The LPA is found to be competitive, at least for small N and away from $D = 2$ where η becomes non-negligible.

The success of large N limit of the LPA to the Polchinski equation is somewhat fortunate. It should be noted that the flow equation (2.68) is not itself exact, but belongs to a large class of equations which yield the correct exponents [23]. Nevertheless, it is an important result and this limit will be the focus of our attention for Fermionic field theory. It is hoped that solving this limit will lead to an increased understanding of the problem and thus help us to deal with finite N theories. Similarly, truncations have been found to be useful, albeit in a limited capacity. In fact from (2.62) we can now see that provided the potential V is real, we expect the eigenvalues, λ_n, to be also real, as claimed. This contrasts with the imaginary exponents which occur using the method of truncations.

Method	ν	ω
Lattice calculation	0.6305	
ϵ-expansion at $O(\epsilon^5)$	0.6310	0.81
Six loop perturbation series	0.6300	0.79
Local potential approximation (Pol)	0.6496	0.6557
LPA Variation method (Pol)	0.6347	0.6093
Local potential approximation (Leg)	0.6604	0.6285
Momentum expansion at $O(p^2)$ (Leg)	0.620	0.898

Table 2.1: Exponents for three dimensional one component Z_2-invariant scalar field theory

The above comparisons are made using results found in [1] and [14].

Appendices

Appendix 2A: Non-trivial fixed points

We include the data for non-trivial fixed points below four dimensions, including the first two subleading fixed points (table 2.2).

D	s_* Wilson-Fisher	s_* 1st subleading	s_* 2nd subleading
4.00	$0.000^\dagger$	-	-
3.90	-0.009	-	-
3.80	-0.021	-	-
3.70	-0.035	-	-
3.60	-0.051	-	-
3.50	-0.070	-	-
3.40	-0.092	-	-
3.30	-0.119	-	-
3.20	-0.149	-	-
3.10	-0.186	-	-
3.00	-0.229	$0.000^\dagger$	-

2.90	-0.280	0.005	-
2.80	-0.342	0.015	-
2.70	-0.417	0.032	-
2.60	-0.512	0.062	-0.001
2.50	-0.632	0.108	-0.008
2.45	-0.707	0.139	-0.015
2.40	-0.793	0.178	-0.028
2.35	-0.895	0.226	-0.049
2.30	-1.019	0.284	-0.082
2.25	-1.172	0.355	-0.137
2.20	-1.368	0.440	-0.226
2.15	-1.633	0.542	-0.378
2.10	-2.027	0.664	-0.654
2.05	-2.738	0.811	-1.249
2.00	$-\infty$	1.000‡	$-\infty$

Table 2.2: Wilson-Fisher and subleading fixed points.

† Fixed points degenerate with the Gaussian.

‡ Fixed points degenerate with the High Temperature.

Appendix 2B: Functional reparameterisation

We can reparameterise the functional, $\mathcal{F}$, in such a manner, that the first term in (2.56) looks like a kinetic term. As a consequence the second term can be interpreted as a time varying potential, unless we deal with two dimensions, when it is static. This is done by making the following change of variables,

$$d\phi = d\tau \; e^{\frac{1}{4}(2-D)\phi^2}. \tag{2.80}$$

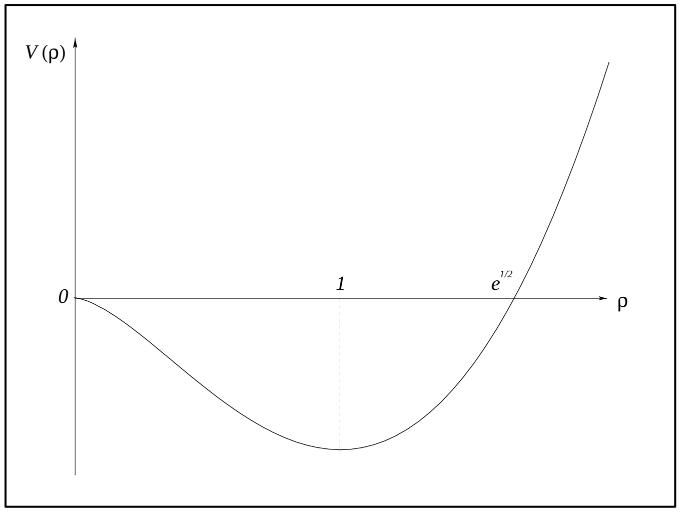

Figure 2.11: Time varying potential in reparameterised functional

Then, replacing $\mathcal{F}$ with $-S$ we arrive at the following, with the desired kinetic term and time varying potential:

$$S = \int d\tau \left(\frac{1}{2}\dot{\rho}^2 - f(\tau)\rho^2 \left(\ln \rho - \frac{1}{2} \right) \right) \tag{2.81}$$

where

$$f(\tau) = \frac{D}{2}\, e^{\frac{1}{2}(2-D)\phi^2} \tag{2.82}$$

and the dot denotes differentiation with respect to τ.

Now we explicitly view (2.81) as an action, and think of a particle at position ρ, at time[9] τ. From (2.82) it is easy to see that in dimension greater than two, $f(\tau)$, and thus the potential, decreases with time, whereas for dimension less than two the reverse is true. The form of the potential,

$$V(\rho) = \rho^2 \left(\ln \rho - \frac{1}{2} \right) \tag{2.83}$$

is shown in figure 2.11.

We now view fixed points as motions in the potential well which are either at rest permanently, or, only come to rest at $\rho = 0$ in an infinite time (recall that in the large ϕ regime, $V_*(\phi) \sim \frac{1}{2}\phi^2$). For example, the Gaussian fixed point corresponds to a particle which simply remains static in the minimum of the well, whereas the High Temperature fixed point corresponds to a solution

[9] not to be confused with the renormalization time defined by (1.29)

which lies at $\rho = \exp(\frac{1}{3}) \approx 1.40$ at $\tau = 0$ and then moves towards $\rho = 0$ as $\tau \to \infty$. The solution for the non-trivial fixed point has been found (figure 2.9), and this corresponds to a particle which lies at $\rho \approx 0.92$ at $\tau = 0$. For this solution the particle oscillates once before coming to rest at $\rho = 0$, again as $\tau \to \infty$. These motions can be deduced directly by observing the behaviour of ρ with ϕ.

Appendix 2C: Gaussian integration

For completeness, we list the set of Gaussian integrals used:

$$I_0 = \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{2.84}$$

found by considering I_0^2 and changing variables, $(x, y) \to (r, \theta)$. Then we define

$$\begin{aligned} I_m &= \int_{-\infty}^{\infty} dx\ x^m e^{-\frac{1}{2}ax^2} && (2.85) \\ &= I_0 \left(\frac{\partial}{\partial b}\right)^m e^{\frac{b^2}{2a}}\bigg|_{b=0} && (2.86) \end{aligned}$$

and calculate

$$\begin{aligned} I_2 &= \frac{1}{a}\sqrt{\frac{2\pi}{a}}, && (2.87) \\ I_4 &= \frac{3}{a^2}\sqrt{\frac{2\pi}{a}}, && (2.88) \\ I_6 &= \frac{15}{a^3}\sqrt{\frac{2\pi}{a}}. && (2.89) \end{aligned}$$

Chapter 3

Zamalodchikov's C-function

In the preceding chapter we argued that the Local Potential Approximation provides competitive estimates of the critical exponents, at least for scalar field theory. The theoretical analysis presented here further vindicates and increases our confidence in the approximation. We begin by introducing conformal field theory, leaving a more detailed analysis to reference [4]. We review Zamalodchikov's C-theorem [24] using the introductory material. The C-theorem, as presented here, only makes sense in two dimensions, although a number of groups have sought to generalise these ideas to other dimensions [25-36]. However, within the Local Potential Approximation we construct a function, C, which has an appropriate generalisation of the Zamalodchikov properties, in general dimension, D. Our C-function 'counts' degrees of freedom at fixed points. We normalise such that it counts one per Gaussian scalar and zero at the High Temperature fixed point (corresponding to an infinitely massive theory).

3.1 The conformal group

As discussed in chapter one, we are interested in non-trivial fixed points which correspond to an interacting scale free theory, known as a conformal field theory. Here we will work mostly in two dimensions, however recent work [22,25] has extended some of the aspects presented to four dimensions. We begin by considering a D-dimensional Riemannian manifold upon which a metric, $g_{\mu\nu}$, is defined. The interval between two points on the manifold is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.1}$$

and a coordinate transformation, $x \to x'$, generates a change in the metric tensor given by

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{3.2}$$

The conformal group is defined to be the subgroup of the transformations that leaves the metric unchanged apart from a non-zero differential scaling function $\Omega(x)$ such that

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{3.3}$$

These metrics are said to be conformally related; the angle between vectors and the ratios of magnitudes of vectors are the same on both manifolds. However, since the length of the vectors differ, this transformation corresponds to a rescaling. The Poincaré group leaves the metric totally invariant and is thus a subgroup of the conformal group with $\Omega = 1$.

3.2 Special properties in two dimensions

The infinitesimal generators of the group can be determined by the analysis of an infinitesimal coordinate transformation, $x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x)$, which causes a line element change. In flat space ($g_{\nu\mu} = \delta_{\nu\mu}$)[1]

$$ds^2 \to ds'^2 \;=\; dx'^\nu dx'_\nu \tag{3.4}$$

[1] working in Euclidean space

$$= \quad ds^2 - (\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)) dx^\mu dx^\nu, \tag{3.5}$$

where we have used (3.2) and defined $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Note that if $\epsilon^\mu(x) = \epsilon^\mu$, *i.e.* a translation, then the line element is invariant under such a transformation. For the transformation to be conformal we require by (3.3) and (3.5) that

$$\Omega(x) ds^2 = ds^2 - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\nu) dx^\mu dx^\nu. \tag{3.6}$$

Then using (3.1) we arrive at

$$\delta_{\mu\nu}(1 - \Omega(x)) dx^\mu dx^\nu = (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu \tag{3.7}$$

and for general dx^μ

$$\delta_{\mu\nu}(1 - \Omega(x)) = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{3.8}$$

Then, using $\delta^{\mu\nu}\delta_{\mu\nu} = \delta^\nu_\nu = D$, we find that the scaling function is given by

$$\Omega(x) = 1 - \frac{2}{D}(\partial.\epsilon). \tag{3.9}$$

where $\partial.\epsilon = \partial^\mu \epsilon_\mu$.

In two dimensions, substituting (3.9) into (3.8) yields the Cauchy-Riemann conditions for the analyticity of $\epsilon_0 + i\epsilon_1$,

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1 \quad \text{and} \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0. \tag{3.10}$$

This motivates us to use complex coordinates defined by

$$z = x^0 + ix^1 \quad \text{and} \quad \bar{z} = x^0 - ix^1, \tag{3.11}$$

and write the infinitesimal transformation as

$$\epsilon(z) = \epsilon_0 + i\epsilon_1 \quad \text{and} \quad \bar{\epsilon}(\bar{z}) = \epsilon_0 - i\epsilon_1. \tag{3.12}$$

Then the conformal transformations coincide with the coordinate transformations

$$z \rightarrow f(z) \quad \text{and} \quad \bar{z} \rightarrow \bar{f}(\bar{z}) \tag{3.13}$$

which have an infinite dimensional local algebra, as $f(z)$ and $\bar{f}(\bar{z})$ can be any analytic functions. These are the generators of the local conformal algebra with Euclidean line element $ds^2 = (dx^0)^2 + (dx^1)^2 = dzd\bar{z}$ transforming as

$$ds^2 \rightarrow \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z} = \Omega dzd\bar{z}. \tag{3.14}$$

However, if we take the transformed coordinates

$$z \rightarrow z' = (1 - \epsilon_n z^n)z \tag{3.15}$$

and

$$\bar{z} \rightarrow \bar{z}' = (1 - \bar{\epsilon}_n \bar{z}^n)\bar{z} \tag{3.16}$$

where n ranges over the integers, then it is easily verified that the infinitesimal generators are given by

$$L_n = -z^{n+1}\frac{\partial}{\partial z} = -z^{n+1}\partial_z \tag{3.17}$$

and

$$\bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{3.18}$$

Now we can construct the algebra:

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{3.19}$$

$$\left[\bar{L}_m, \bar{L}_n\right] = (m-n)\bar{L}_{m+n} \tag{3.20}$$

and

$$\left[L_m, \bar{L}_n\right] = 0. \tag{3.21}$$

The latter condition, (3.21), shows that the local conformal algebra is a direct sum of two independent subalgebras. The action of the $D = 2$ conformal group thus factorises into independent actions on z and $\bar{z}$ and the Greens functions of the theory are such that z and $\bar{z}$ can be treated as independent variables.

3.3 The stress tensor as the generator of scaling

Let us now consider a Euclidean action $S(\phi)$ depending only on the field and its first partial derivatives, invariant under translation (associated with momentum conservation), rotation (angular momentum conservation) and dilation. Let us perform an infinitesimal change of variables:

$$x_\mu \to x'_\mu = x_\mu - \epsilon_\mu(x). \tag{3.22}$$

As discussed in Appendix 3A, due to translational invariance, the variation of the action involves only the partial derivatives of $\epsilon_\mu(x)$:

$$\delta S(\phi) = \int d^D x\ T_{\mu\nu}(x) \partial_\mu \epsilon_\nu(x) \tag{3.23}$$

where $T_{\mu\nu}(x)$ is the stress tensor. Imposing other conformal symmetries, we deduce some required properties of the stress tensor [1]. For example rotation invariance implies δS vanishes for

$$\epsilon_\mu = \Lambda_{\mu\nu} x^\nu \tag{3.24}$$

in which $\Lambda_{\mu\nu}$ is an arbitrary antisymmetric matrix. Therefore the stress tensor must be symmetric:

$$T_{\mu\nu} = T_{\nu\mu}. \tag{3.25}$$

Similarly, dilation invariance corresponds to

$$\epsilon_\mu = \lambda x_\mu \tag{3.26}$$

and implies the vanishing of the trace of the stress tensor:

$$T_{\mu\mu} \equiv \Theta = 0. \tag{3.27}$$

In complex coordinates we have $\partial_{\bar{z}} z = \partial_z \bar{z} = 0$ as z and $\bar{z}$ are treated independent. We refer the metric to the complex coordinate frame using (3.2),

$$g = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{3.28}$$

and similarly calculate the stress tensor,

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}), \tag{3.29}$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \tag{3.30}$$

and

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}T_{\mu\mu}. \tag{3.31}$$

Furthermore, we can calculate the derivatives of (3.29) and (3.30) with respect to $\bar{z}$ and z respectively, such that by conservation of the stress tensor (see Appendix 3A), $\partial_\mu T_{\mu\nu} = 0$, we deduce

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = \partial_{\bar{z}} T + \frac{1}{4}\partial_z \Theta = 0 \tag{3.32}$$

and similarly

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = \partial_z \bar{T} + \frac{1}{4}\partial_{\bar{z}} \Theta = 0. \tag{3.33}$$

Here we have defined $T(z,\bar{z}) \equiv T_{zz}(z,\bar{z})$ and $\bar{T}(z,\bar{z}) \equiv T_{\bar{z}\bar{z}}(z,\bar{z})$, Θ is given by (3.27) and we have used (3.31). Finally we note that for a scale invariant field theory[2] (3.31) vanishes by (3.27) and we find $\partial_{\bar{z}} T = \partial_z \bar{T} = 0$ which implies that the stress tensor factorises into independent pieces, $T(z,\bar{z}) = T(z)$ and $\bar{T}(z,\bar{z}) = \bar{T}(\bar{z})$.

3.4 Conformal weights

In general, the fields in a conformal theory obey the two dimensional equivalent of the tensor transformation law,

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi\left(f(z), \bar{f}(\bar{z})\right) \tag{3.34}$$

[2]corresponding to a fixed point

where $(h, \bar{h})$ are the conformal weights of Φ. We observe that $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$ have conformal weights (2,0) and (0,2) respectively. Similarly the trace of the stress tensor, $\Theta = 4T_{z\bar{z}}$, has conformal weight (1,1). Now consider two fields $\Phi_1(z, \bar{z})$ and $\Phi_2(z, \bar{z})$ with conformal weights $(h_1, \bar{h}_1)$ and $(h_2, \bar{h}_2)$ respectively. Then by (3.34) $< \Phi_1(z, \bar{z})\Phi_2(0, 0) > \rightarrow a^{(h_1+h_2)}\bar{a}^{(\bar{h}_1+\bar{h}_2)} < \Phi_1(f(z), \bar{f}(\bar{z}))\Phi_2(0, 0) >$ for $f(z) = az$. For a rotation $a = e^{i\theta}$, and thus we require

$$< \Phi_1(z, \bar{z})\Phi_2(0, 0) >= \frac{E(z\bar{z})}{z^{h_1+h_2}\bar{z}^{\bar{h}_1+\bar{h}_2}} \tag{3.35}$$

for the transformation to have no effect. In (3.35) E is an arbitrary scalar function. If, in addition we have scale invariance (corresponding to a fixed point), *i.e.* $a = re^{i\theta}$, we have a stronger condition for invariance,

$$< \Phi_1(z, \bar{z})\Phi_2(0, 0) >= \frac{E}{z^{h_1+h_2}\bar{z}^{\bar{h}_1+\bar{h}_2}} \tag{3.36}$$

where now E must be a constant.

3.5 The operator product expansion

To generate the operator product expansion we construct a conserved charge on the conformal z-plane,

$$Q = \frac{1}{2\pi i}\int \left(dzT(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right), \tag{3.37}$$

associated with the conserved stress tensor discussed above. For the full conformal group there are an infinite set of conserved currents (associated with scale invariance at a fixed point), $\epsilon(z)T(z)$ and their antiholomorphic partners. These charges can then be used to generate an infinitesimal symmetry variation in the field, Φ, given by

$$\begin{aligned}
\delta_{\epsilon,\bar{\epsilon}}\Phi(w, \bar{w}) &= \frac{1}{2\pi i}\int [dzT(z)\epsilon(z), \Phi(w, \bar{w})] \\
&+ \frac{1}{2\pi i}\int [d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \Phi(w, \bar{w})] \qquad (3.38)\\
&= \frac{1}{2\pi i}\oint_w dz\epsilon(z)T(z)\Phi(w, \bar{w}) \\
&+ \frac{1}{2\pi i}\oint_{\bar{w}} d\bar{z}\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\Phi(w, \bar{w}) \qquad (3.39)
\end{aligned}$$

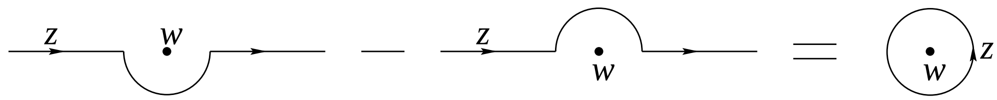

Figure 3.1: Evaluation of the commutator (3.38) on the conformal plane

where we have combined the two z integrations in the commutator into a single contour integral around w [4] (figure 3.1) and normalised with a factor $\frac{1}{2\pi i}$. However, taking[3] $f(z) = z + \epsilon(z)$ and using (3.34) we can expand for small ϵ to find

$$\begin{aligned} \Phi(w,\bar{w}) &\rightarrow (1 + h\partial_z\epsilon)(1 + \bar{h}\partial_{\bar{z}}\bar{\epsilon})(\Phi + \epsilon\partial_z\Phi + \bar{\epsilon}\partial_{\bar{z}}\Phi) \quad (3.40) \\ &= \Phi + h(\partial_z\epsilon)\Phi + \bar{h}(\partial_{\bar{z}}\bar{\epsilon})\Phi + \epsilon\partial_z\Phi + \bar{\epsilon}\partial_{\bar{z}}\Phi. \quad (3.41) \end{aligned}$$

Then, for (3.39) to be consistent with (3.41) we deduce that we must have

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \cdots \quad (3.42)$$

and similarly

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \cdots \quad (3.43)$$

where the dots contain contributions analytic in z. We may verify (3.42) and (3.43) by substituting into (3.39) and performing the contour integration, reproducing the variation in (3.41). The expressions (3.42) and (3.43) are known as the short distance operator expansion and apply generally to a tensor of weight $(h, \bar{h})$.

Alternatively, in the corresponding quantum theory, $T(z)$ is written as a normal ordered product of two fields and Wicks rules can be used to reproduce the operator expansions above. Then, it is found that in the case of the expansion of T with itself an additional term arises due to double Wick contractions, which can be written [4]

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)^2}\partial_w T(w) + \cdots \quad (3.44)$$

where c is known as the conformal anomaly[4] [37], and the coefficient of $T(w)$ reflects the fact

[3] consistent with the transformation used in section 3.2 for small $\epsilon(z) = -z^{n+1}$

[4] also known as the Virasoro central charge

that T has a conformal weight $h = 2$. A similar expression for the expansion of $\bar{T}$ with itself can be found, however we will now concentrate purely on the unbarred case. Observe that as the stress tensor written as a normal ordered product annihilates the vacuum, the two-point function is given by

$$< T(z)T(0) >= \frac{c}{2z^4}, \tag{3.45}$$

and thus it is expected that $c \geq 0$ for a unitary theory. Note that this is precisely the form required at a fixed point by (3.36).

3.6 The Virasoro algebra

To generate the Virasoro algebra [4] it is useful to define a Laurent expansion of the stress tensor in terms of operator modes,

$$T(z) = \sum_{n \epsilon Z} z^{-n-2} L_n \tag{3.46}$$

with

$$L_n = -\frac{1}{2\pi i} \oint dz z^{n+1} T(z). \tag{3.47}$$

Classically these are simply Fourier modes but become operators modes in the context of quantum field theory. From (3.15) we substitute $\epsilon(z) = -z^{n+1}$ into (3.39) to verify that (3.47) is indeed the infinitesimal generator. The exponent, z^{-n-2}, is chosen to give the operators, L_n, a scaling dimension of n under a scaling $z \to az$, consistent with (3.18), since then $T(z) \to \frac{1}{a^2} T(az)$ by (3.44) and thus $L_n \to a^n L_n$ in (3.47).

To derive the algebra satisfying the above modes we must commute the operators L_n to give a double contour integral,

$$[L_n, L_m] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{n+1} T(z) w^{m+1} T(w). \tag{3.48}$$

The z contour integration can then be performed by initially fixing w and deforming the difference between the two z integrations in (3.48) into a single contour around w (figure 3.2). If we then use the stress tensor self product, (3.44), and finally integrate over w we can write the

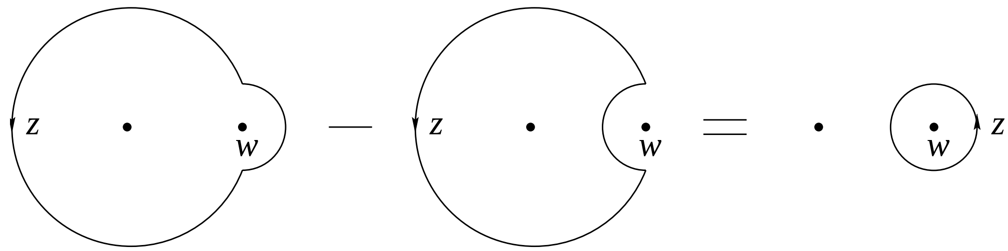

Figure 3.2: Evaluation of the commutator (3.48) on the conformal plane

commutator as,

$$\begin{aligned} [L_n, L_m] &= \oint \frac{dw}{2\pi i} w^{m+1} \oint_w \frac{dz}{2\pi i} z^{n+1} \Big(\frac{c}{2(z-w)^4} \\ &+ \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)^2} \partial_w T(w) + \cdots \Big) \qquad (3.49) \\ &= \oint \frac{dw}{2\pi i} w^{m+1} \Big(\frac{c}{12}(n+1)n(n-1) w^{n-2} \\ &+ 2(n+1) w^n T(w) + w^{n+1} \partial_w T(w) + \cdots \Big). \qquad (3.50) \end{aligned}$$

where we have explicitly performed the z integration. Re-expressing the third integral in (3.50) by parts and combining it with the second term, we finally arrive at

$$[L_n, L_m] = \frac{c}{12}(n^3 - n) \oint \frac{dw}{2\pi i} w^{n+m-1} + (n-m) \oint \frac{dw}{2\pi i} w^{n+m+1} T(w). \tag{3.51}$$

Referring to (3.47) we identify the second integral as L_{m+n} and then performing the first integral we derive the Virasoro algebra,

$$[L_n, L_m] = \frac{c}{12}(n^3 - n)\delta_{n+m,0} + (n-m)L_{n+m} \tag{3.52}$$

and similarly

$$[\bar{L}_n, \bar{L}_m] = \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0} + (n-m)\bar{L}_{n+m}. \tag{3.53}$$

Every conformally invariant quantum field theory has a particular representation of this algebra with some value of c or $\bar{c}$. We observe that the Virasoro algebra reduces to the classical case for $c = \bar{c} = 0$, (3.19) and (3.20).

3.7 Counting

In the following sections of this chapter we motivate, review, and discuss the implications of Zamalodchikov's celebrated C-theorem [24] for two dimensional quantum field theory, and establish three important properties for his C-function. The third property depends on the conformal anomaly and thus only makes sense in two dimensions. A number of groups have sought to generalise these ideas to higher dimensions [25]. Then, within the LPA, we will display a C-function which has the first two of these properties and an appropriate generalisation of the third property in any dimension D. At fixed points, our C-function is extensive, videlicet additive in mutually non-interacting degrees of freedom, as is also true of the conformal anomaly.

Suppose that the fields of the theory form two non-interacting sets. Let us write $\phi_a = \phi_a^{(1)}$ when the field belongs to the first set, and $\phi_a = \phi_a^{(2)}$ when it belongs to the second set. Similarly, as they are non-interacting, the couplings g split into two sets denoted $g^{(1)}$ and $g^{(2)}$ and the action can be written

$$S[\phi, t] = S^{(1)}[\phi^{(1)}, t] + S^{(2)}[\phi^{(2)}, t]. \tag{3.54}$$

Then, the variation in the action and thus the stress tensor, can be written as a sum of two separate parts as can be seen from (3.23). Furthermore, the generators given by (3.47) separate into two independent pieces and thus the Virasoro algebra in (3.52) leads to the conclusion that c is extensive at fixed points, *i.e.*

$$c(g_*) = c(g_*^{(1)}) + c(g_*^{(2)}). \tag{3.55}$$

This counting will be the third property of our C-function derived below.

The interpretation given to this is one of counting degrees of freedom. At the Gaussian fixed point, $V_*^G = 0$, the conformal anomaly counts one degree of freedom per scalar. At the High Temperature fixed point, $V_*^{HT} = \frac{1}{2}\phi^2 - \frac{N}{D}$, the conformal anomaly vanishes, corresponding to an infinitely massive theory[5] with no degrees of freedom. Thus, we will normalise such that our C counts one for each Gaussian scalar and zero at the High Temperature fixed point.

[5] in units of Λ

3.8 Zamalodchikov's C-theorem

Zamalodchikov's C-thereom provides an explicit geometric framework for the space of two dimensional quantum field theories [21,22] and demonstrates the irreversibility of renormalization group flows, thus proving that exotic flows are missing. Zamalodchikov established three properties for his C-function,

I) There exists a function $C(g) \geq 0$ of such a nature that

$$\frac{dC}{dt} \equiv \beta^i(g) \frac{\partial C(g)}{\partial g^i} \leq 0 \tag{3.56}$$

where g^i form an infinite set of parameters (the coupling constants discussed in section 1.10) and the beta functions are defined by (1.41).

II) $C(g)$ is stationary only at the fixed points of the renormalization group, *i.e.* when $g(t) = g_*$.

III) The value of $C(g)$ at the fixed point g_* is the same as the corresponding conformal anomaly [37]. (This property thus only makes sense in two dimensions.)

The proof relies on rotational invariance, reflection positivity and the conservation of the stress tensor (a property which is a general consequence of translational invariance and is also valid away from the critical point). Consider some point on a renormalization group trajectory (refer to figure 1.4) specified by a set of couplings, g^i. For the time being we suppress the dependence on g^i. As discussed, away from the fixed point the Hamiltonian is no longer invariant under dilations and in addition to the components T and $\bar{T}$ the stress tensor has non-zero trace, Θ. These components have conformal weights (2,0), (0,2) and (1,1) respectively, so that by (3.35) we can write their two-point functions as

$$\begin{aligned}
&& < T(z,\bar{z})T(0,0) > &= \frac{F(z\bar{z})}{z^4}, & (3.57)\\
< \Theta(z,\bar{z})T(0,0) > &= & < T(z,\bar{z})\Theta(0,0) > &= \frac{G(z\bar{z})}{z^3\bar{z}}, & (3.58)\\
&& < \Theta(z,\bar{z})\Theta(0,0) > &= \frac{H(z\bar{z})}{z^2\bar{z}^2}. & (3.59)
\end{aligned}$$

where F, G and H are non-trivial scalar functions. Taking the correlation function of (3.32) with $T(0,0)$ and $\Theta(0,0)$ yields two equations

$$\begin{aligned} G &= \frac{1}{3}(\dot{G}+4\dot{F}) && (3.60) \\ G &= \dot{G}+\frac{1}{4}(\dot{H}-2H) && (3.61) \end{aligned}$$

where we define $\dot{F} = z\bar{z}F'(z\bar{z})$. Then, defining

$$C = 2F - G - \frac{3}{8}H \tag{3.62}$$

and eliminating G from (3.60) and (3.61) yields

$$\dot{C} = -\frac{3}{4}H. \tag{3.63}$$

Reflection positivity requires that $< \Theta\Theta > \geq 0$, so that by (3.59) $H \geq 0$. Then from (3.63) $\dot{C} \leq 0$ which implies $C' \leq 0$ as $z\bar{z} \geq 0$. This in turn implies that C is a non-increasing function of $r = (z\bar{z})^{\frac{1}{2}} = ((x^0)^2 + (x^1)^2)^{\frac{1}{2}} \geq 0$ as $C' = 2r\frac{\partial C}{\partial r}$.

Now imagine making a renormalization group transformation such that the lattice spacing $a \to a + \delta a$. Since our C-function is dimensionless, this is equivalent to sending $r \to r - \delta r$, and the coupling constants g^i will flow according to the renormalization group equation,

$$\left(\beta_i(g)\frac{\partial}{\partial g^i} - r\frac{\partial}{\partial r}\right) C(r, g^i) = 0. \tag{3.64}$$

If we now define $C(g) \equiv C(r_o, g^i)$, where r_o is some arbitrary but fixed length scale, then given that $\frac{\partial C}{\partial r} \leq 0$, we deduce the first Zamalodchikov property for C. Moreover, C is stationary if and only if $H = 0$, which by reflection positivity, implies $\Theta = 0$, so that the theory is scale invariant (condition for (3.27)) and thus corresponds to a fixed point, which implies property two. Finally at such a fixed point, the vanishing of the trace of the stress tensor (Θ) by (3.27) implies that $G = H = 0$ by (3.58) and (3.59). Thus (3.62) becomes $F = \frac{1}{2}C$, so that comparing (3.45) with (3.57) shows that $C = c$ verifying the final Zamalodchikov property.

3.9 M_{ij} as a metric

We begin by noticing that M_{ij}, defined by (2.60), can be interpreted as a positive definite metric [22] in the space of quantum field theories. Initially we consider the interval,

$$\begin{aligned} ds^2 &= M_{ij}(g)dg^i dg^j \quad (3.65)\\ &= a^N \int d^N\phi\, G \left(dg^i \frac{\partial \rho}{\partial g^i}\right)^2 \quad (3.66)\\ &\geq 0. \quad (3.67)\end{aligned}$$

The final observation (positive definiteness) holds providing V is real (corresponding to a unitary theory in Minkowski space). The equality in (3.67) is only reached if $dg^i = 0$. Perhaps more importantly we observe that the interval is indeed invariant under a change of coordinates,

$$\begin{aligned} ds^2 &= M'_{ij}(g')dg'^i dg'^j \quad (3.68)\\ &= M'_{ij}(g')\frac{\partial g'^i}{\partial g^k}\frac{\partial g'^j}{\partial g^l}dg^k dg^l \quad (3.69)\\ &= M_{kl}dg^k dg^l. \quad (3.70)\end{aligned}$$

3.10 A C-function representation of the LPA

We define our $C(g)$-function through

$$\mathcal{F} = \frac{Db^C}{4} \quad (3.71)$$

where $b > 1$ and $\mathcal{F}$ is defined by (2.56). This C-function satisfies our appropriate generalisation of Zamalodchikov's properties in general dimension, D. Generally

$$\frac{\partial b^C}{\partial g^i} = \frac{\partial C}{\partial g^i} b^C \ln(b). \quad (3.72)$$

However, by differentiating (3.71) with respect to g^i and using (2.59) we find

$$\frac{\partial b^C}{\partial g^i} = -\frac{4}{D} M_{ij}\beta^j \quad (3.73)$$

and therefore

$$\frac{\partial C}{\partial g^i} = -\hat{M}_{ij}\beta^j \tag{3.74}$$

where we have defined $\hat{M}_{ij}$ through $M_{ij} = (\mathcal{F}\ln b)\hat{M}_{ij}$. Thus, we find that

$$\begin{aligned} \frac{dC}{dt} &= \beta^i \frac{\partial C}{\partial g^i} && (3.75) \\ &= -\beta^i \hat{M}_{ij}\beta^j. && (3.76) \end{aligned}$$

Using an argument similar to that used to derive (3.67) we can now write

$$\begin{aligned} \frac{dC}{dt} &= a^N \int d^N\phi \; G \left(\beta^i \frac{\partial \rho}{\partial g^i}\right)^2 && (3.77) \\ &\leq 0 && (3.78) \end{aligned}$$

confirming the first Zamalodchikov property. Given that the β-functions vanish at a fixed point, the second property follows immediately from (3.74).

The appropriate generalisation of the third property is found by supposing that the fields form two mutually non-interacting sets. Using our previous notation we write $\phi_a = \phi_a^{(1)}$ when the field belongs to the first set, and $\phi_a = \phi_a^{(2)}$ when it belongs to the second set, such that the couplings g also split into two sets denoted $g^{(1)}$ and $g^{(2)}$ and by (3.54) the potential can be written

$$V(\phi, t) = V^{(1)}(\phi^{(1)}, t) + V^{(2)}(\phi^{(2)}, t) \tag{3.79}$$

and thus ρ factorises: $\rho = \rho^{(1)}\rho^{(2)}$. However, substituting the fixed point equation $\frac{\delta \mathcal{F}}{\delta \rho} = 0$ into (2.56), we derive that fixed points $\rho(\phi, t) = \rho_*(\phi)$ satisfy

$$\mathcal{F}[\rho_*] = \frac{D}{4} a^N \int d^N\phi \; G\rho_*^2 \tag{3.80}$$

and we immediately notice that $\mathcal{F}[\rho_*] = \frac{4}{D}\mathcal{F}[\rho_*^{(1)}]\mathcal{F}[\rho_*^{(2)}]$. Thus from (3.71) we observe that at fixed points our C is extensive, as required by (3.55).

In two dimensions, at the Gaussian fixed point, the conformal anomaly counts one degree of freedom per scalar, *i.e.* is equal to N, and at the High Temperature fixed point, it vanishes. We will take this to be true in general dimension D. Thus the normalisation factors a and b

can be uniquely determined by requiring that our C agree with this counting in any dimension D. Substituting $C = 0$ in (3.71), and $\rho_* = \exp(-V_*^{HT})$ in (3.80) and performing the Gaussian integral yields

$$a = e^{2/D}\sqrt{\frac{D+2}{4\pi}}. \tag{3.81}$$

Similarly, for the Gaussian fixed point we substitute $C = N$ in (3.71) and $\rho_* = \exp(-V_*^G) = 1$ in (3.80) and find

$$b = e^{2/D}\sqrt{\frac{D+2}{D-2}}. \tag{3.82}$$

Note that as required, $b > 1$, at least for $D \geq 2$.

3.11 Examples

Consider the simple example of the Gaussian fixed point perturbed by the mass operator, for a single scalar field. Thus we set $V(\phi, t) = g^1(t) + \frac{1}{2}g^2(t)\phi^2$. Then the β-functions are given by (2.39) and (2.40). Integrating (2.40) yields

$$g^2(t) = \frac{1}{1 + ke^{-2t}} \tag{3.83}$$

and then (2.39) yields

$$\begin{aligned} g^1(t) &= g_0^1 e^{Dt} + e^{Dt}\int \frac{e^{-D\tau}}{1+ke^{-2\tau}}d\tau & (3.84)\\ &= g_0^1 e^{Dt} - \frac{1}{2}\int_0^1 \frac{u^{D/2-1}}{1+uke^{-2t}}du, & (3.85) \end{aligned}$$

where $u = e^{2t-2\tau}$ and k and $g_0^1 = g^1(0)$ are constants of integration. Considering the limit $t \to -\infty$, we see that this solution indeed emanates from the Gaussian fixed point. We normalised the special solution in (3.85) so that with $g_0^1 = 0$, it tends to V_*^{HT} as $t \to +\infty$.

Since ρ is Gaussian, $\mathcal{F}$ is readily determined using (2.56),

$$\mathcal{F} = \frac{a}{2}e^{-2g^1}\left(2g^1 D + \frac{4(g^2)^2 + 6g^2 D - 2D + D^2}{D - 2 + 4g^2}\right)\sqrt{\frac{\pi}{D-2+4g^2}}. \tag{3.86}$$

In particular, using (3.81) we verify

$$\mathcal{F}^G = \frac{D}{4} e^{2/D} \sqrt{\frac{D+2}{D-2}} \tag{3.87}$$

and

$$\mathcal{F}^{HT} = \frac{D}{4} \tag{3.88}$$

for the Gaussian ($g^1 = g^2 = 0$) and High Temperature ($g^1 = -\frac{1}{D}$ and $g^2 = 1$) respectively. Then using (3.71) and (3.82) we deduce that $C^G = 1$ and $C^{HT} = 0$ as expected. Indeed we may further verify that when $g_0^1 = 0$, $C(t)$ flows from 1 to 0 as t runs from $-\infty$ to $+\infty$. Note that if $g_0^1 \neq 0$, then $C(t) \to \pm\infty$ as $t \to \infty$, depending on the sign of g_0^1. This seems in contradiction with the idea that C counts degrees of freedom, since evidently the vacuum energy should not figure *per se* in this counting. However, we recall from (3.55) that C is extensive (*i.e.* counts) only at fixed points, and with $g_0^1 \neq 0$ the system never reaches another fixed point as $t \to \infty$.

From the Gaussian form of G, we recognise that at both the Gaussian and High Temperature fixed points, the metric (2.60) is diagonalised by choosing the operators $\mathcal{O}_i = \frac{\partial V}{\partial g^i}$ to be products of Hermite polynomials H_n in the ϕ_a. Since these also turn out to diagonalise $\frac{\partial^2 \mathcal{F}}{\partial g^i \partial g^j}(g_*)$, they are the eigenperturbations, and the corresponding eigenvalues follow straightforwardly. Choosing the Gaussian fixed point $\rho_* = 1$ for example, and again specialising to the case of a single scalar field for simplicity, we thus take $\mathcal{O}_n = H_n(\frac{\phi}{2}\sqrt{D-2})$, $n = 0, 1, \cdots$. From (2.56) we then obtain

$$\frac{\partial^2 \mathcal{F}}{\partial g^i \partial g^j} = -DM_{ij} + a \int d\phi \; G \frac{\partial \mathcal{O}_i}{\partial \phi} \frac{\partial \mathcal{O}_j}{\partial \phi}. \tag{3.89}$$

However, using $H_n' = 2nH_{n-1}$ where prime denotes differentiation with respect to the argument, we deduce $\frac{\partial \mathcal{O}_n}{\partial \phi} = n\sqrt{D-2}\mathcal{O}_{n-1}$ so that (3.89) yields

$$\frac{\partial^2 \mathcal{F}}{\partial g^i \partial g^j} = -DM_{ij} + ij(D-2)M_{i-1\ j-1}. \tag{3.90}$$

Then, given that the metric has non-zero components [38] $M_{nn} = 2^{n+1} n! a \sqrt{\frac{\pi}{D-2}}$, from (2.62) we recover the Gaussian spectrum of eigenvalues,

$$\lambda_n \quad = \quad -\frac{1}{M_{nn}} \frac{\partial^2 \mathcal{F}}{\partial g^n \partial g^n} \tag{3.91}$$

$$= D - \frac{1}{2}(D-2)n \tag{3.92}$$

as expected. The High Temperature fixed point can be solved in a similar manner.

3.12 Discussion

The C-theorem has the interpretation that renormalization group flows go 'downhill'. As described it rules out the existence of exotic flows and in particular restricts the possible fixed points to which unstable directions at a given fixed point may flow. Various authors [39] have reported on the appealing interpretation of the C-function as a kind of entropy of information about the critical system. Under renormalization, information is lost about the short distance behaviour of the correlation functions, corresponding to a decreasing C.

The value of the C-function at a fixed point is interpreted as the number of degrees of freedom, which may initially seem counter intuitive given that the C-function we have defined (3.71) is non-integer at the Wilson-Fisher fixed point[6]. However, the following example may convince us that this is in fact not as unusual as we might imagine. Consider a ball attached to a solid wall on a spring. Initially the ball might have one degree of freedom (position in one dimension). Then imagine increasing the spring stiffness in a smooth manner until eventually the ball is permanently stationary. The degrees of freedom of the ball have reduced smoothly from one to zero.

It is probably not possible within the Local Potential Approximation to establish a more concrete link between our C-function and Zamalodchikov's C. Of course, it would be very interesting to understand if these observations generalise to higher orders in the derivative expansion [40] (which likely would allow a direct comparison with Zamalodchikov's C), or indeed generalise to an exact expression along the present lines.

[6]the exact LPA solution yields $c = 0.9867$ whereas variation yields $c = 0.9896$

Appendices

Appendix 3A: The stress tensor

Generally, if the Lagrangian density $\mathcal{L}$ depends only on the field $\phi(x)$ and its derivative $\partial_\mu\phi(x)$, the variation in the Lagrangian can be written

$$\begin{aligned} \delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi(x)}\delta\phi(x) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(x))}\delta\partial_\mu\phi(x) && (3.93) \\ &= \left(\frac{\delta\mathcal{L}}{\delta\phi(x)} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(x))}\right)\delta\phi(x) + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(x))}\delta\phi(x)\right) && (3.94) \end{aligned}$$

where we have used

$$\delta\partial_\mu\phi(x) = \partial_\mu\delta\phi(x). \tag{3.95}$$

To derive the equations of motion we assume that the arbitrary variations, $\delta\phi(x)$, vanish at infinity so that the last term in (3.94) integrates to zero in the variation of the action. However we suppose that $\phi(x)$ satisfies the equations of motion and take $\delta\phi(x)$ not as an arbitrary variation, but a translation:

$$\delta\phi(x) = -\epsilon^\nu\partial_\nu\phi(x). \tag{3.96}$$

Then the first bracket in (3.94) vanishes and substituting (3.96) we find

$$\delta\mathcal{L} = -\epsilon^\nu\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(x))}\partial_\nu\phi(x)\right). \tag{3.97}$$

However, $\mathcal{L}$ is not invariant under a global transformation, rather we have

$$\delta\mathcal{L} = -\epsilon^\mu\partial_\mu\mathcal{L}. \tag{3.98}$$

Thus, from (3.97) and (3.98)

$$\epsilon^\nu \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) = 0 \tag{3.99}$$

which, since it is true for all ϵ^ν, implies

$$\partial^\mu T_{\mu\nu} = 0 \tag{3.100}$$

where we have defined

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \tag{3.101}$$

The stress tensor, $T_{\mu\nu}$, is the conserved quantity corresponding to translational invariance.

Now consider the variation of the action with a local translation $\epsilon^\nu(x)$,

$$\begin{aligned} \delta S &= \int d^D x \delta \mathcal{L} && (3.102) \\ &= -\int d^D x \left(\frac{\delta \mathcal{L}}{\delta \phi(x)} \epsilon^\nu(x) \partial_\nu \phi(x) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \partial_\mu (\epsilon^\nu(x) \partial_\nu \phi(x)) \right) && (3.103) \\ &= -\int d^D x \left(\epsilon^\nu(x) \left(\frac{\delta \mathcal{L}}{\delta \phi(x)} \partial_\nu \phi(x) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \partial_\mu \partial_\nu \phi(x) \right) \right. && (3.104) \\ &+ \left. \partial_\mu \epsilon^\nu(x) \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \partial_\nu \phi(x) \right) \end{aligned}$$

where we have used (3.95) and (3.96). Replacing the inner brackets of the first term in (3.104) with $\partial_\nu \mathcal{L}$ and integrating the last term by parts we finally arrive at

$$\begin{aligned} \delta S &= \int d^D x \epsilon^\nu(x) \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \partial_\nu \phi(x) - \delta^\mu_\nu \mathcal{L} \right) && (3.105) \\ &= \int d^D x \epsilon^\nu(x) \partial^\mu T_{\mu\nu} && (3.106) \end{aligned}$$

by the definition of the stress tensor (3.101).

Chapter 4

Fermionic field theory

In the previous two chapters we have assembled a considerable portfolio of evidence in support of the Local Potential Approximation. However, it must be admitted that (here and in general) most success has arisen from focusing on purely Bosonic field theories, and ultimately we must extend these constructions to include Fermions in a reliable manner. Here we will be specifically interested in the Fermionic Polchinski and Legendre flow equations. For a finite number of components, the LPA to the Legendre flow equation is significantly more difficult to implement than for the scalars and the Polchinski flow equation will form the focus of our attention. However, a technical problem involving chiral symmetry emerges and we find that the Polchinski LPA is trivial. These results are compared with exact solutions obtained in the large N limit using the Fermionic Legendre flow equation[1]. As a result we develop an understanding of the Polchinski LPA for a finite number of components, which are presented towards the end of this chapter.

4.1 Introduction

In this chapter we will be interested in Fermionic field theory, and in particular spinors, which we label $\psi_{\alpha a}$, where here α runs over internal spinor indices from 1 to N_i and a is the flavour

[1] only the Legendre flow equation is exact in the large N limit [23]

index running from 1 to N_f. These satisfy the well known anti-commutation relations including

$$\{\psi_{\alpha a}, \psi_{\beta b}\} = 0 \;\; \text{and} \;\; \{\psi_{\alpha a}, {\psi^{\dagger\beta}}_b\} = \hbar {\delta_\alpha}^\beta \delta_{ab}\delta(x-y). \tag{4.1}$$

This so called Grassmann nature implies that $\psi_i\psi_i = 0$ where i runs over all indices, and thus that a function of a finite number of Fermion fields can be expanded as an exact finite series. It is useful to use the traceless gamma matrices [1] γ^μ $(\mu = 0, \cdots, 3)$ and $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, which satisfy $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^0)^2 = I$ where I is the unit matrix. The adjoint spinor is then defined by $\bar{\psi}^\alpha_a = \psi^{\dagger\beta}_a {\gamma^0}_\beta{}^\alpha$. We will be using derivatives $\frac{\delta}{\delta\bar{\psi}}$ and $\frac{\delta}{\delta\psi}$ which both act on the left in the conventional manner.

A theory is said to be chirally symmetric if it is unchanged under

$$\psi \to \gamma_5\psi \tag{4.2}$$

and

$$\bar{\psi} \to \bar{\psi}\gamma_5. \tag{4.3}$$

In particular we observe that a chirally invariant theory has no mass term, $\bar{\psi}\psi$, or indeed any odd power of $\bar{\psi}\psi$. However the kinetic term $\bar{\psi}\not{q}\psi$ is acceptable, where we define $\not{q} = \gamma^\mu q_\mu$. Ultimately we are interested in the Gross-Neveu model [41] which can be defined by the bare Lagrangian

$$\mathcal{L} = \bar{\psi}^\alpha_a(x){\not{\partial}_\alpha}^\beta\psi_{\beta a}(x) + g(\bar{\psi}^\alpha_a(x)\psi_{\alpha a}(x))^2 \tag{4.4}$$

which is seen to be chirally invariant.

4.2 Conditions on Fermionic potentials

To derive the condition that must be satisfied by the Fermionic potential we impose that the Minkowski Hamiltonian density,

$$\mathcal{H}[\psi(x), \bar{\psi}(x)] = \dot{\psi}(x)_{\alpha a}\Pi^\alpha_a(x) - \mathcal{L}_M[\psi(x), \bar{\psi}(x)] \tag{4.5}$$

where the Lagrangian is given by

$$\mathcal{L}_M[\psi(x),\bar{\psi}(x)] = i\bar{\psi}_a^{\alpha}(x)\partial\!\!\!/_{\alpha}{}^{\beta}\psi_{\beta a}(x) - V(\psi^{\dagger}(x)\gamma_0\psi(x)) \tag{4.6}$$

and

$$\Pi_a^{\alpha}(x) = \frac{\partial \mathcal{L}_M}{\partial \dot{\psi}_{\alpha a}(x)} = -i\psi_a^{\dagger\alpha}(x), \tag{4.7}$$

must be positive definite. For convenience we suppress the indices on the arguments of functions. The dots in (4.5) and (4.7) denote differentiation with respect to $t = x^0$. By substituting (4.6) and (4.7) into (4.5) we find an alternative expression for the Hamiltonian density

$$\mathcal{H}[\psi(x),\bar{\psi}(x)] = \psi_a^{\dagger\alpha}(x)\,(\mathbf{a}.\mathbf{p})_{\alpha}{}^{\beta}\,\psi(x)_{\beta a} + V(\psi^{\dagger}(x)\gamma_0\psi(x)) \tag{4.8}$$

where $a^i = \gamma^0\gamma^i$ and $p^i = -i\partial^i$.

We begin by substituting this Hamiltonian into the equation of motion

$$i\dot{\psi}_{\alpha a}(x) = \left[\psi_{\alpha a}(x), \int d^D y \mathcal{H}[\psi(y),\bar{\psi}(y)]\,\right] \tag{4.9}$$

such that using (4.1) we can perform the integrals over delta functions, yielding

$$i\dot{\psi}_{\alpha a}(x) = (\mathbf{a}.\mathbf{p})_{\alpha}{}^{\beta}\;\psi_{\beta a}(x) + \left[\psi_{\alpha a}(x), \int d^D y V(\psi^{\dagger}(y)\gamma^0\psi(y))\right]. \tag{4.10}$$

Comparing this with (4.8) we observe that

$$\begin{aligned}\mathcal{H}[\psi(x),\bar{\psi}(x)] = i\psi_a^{\dagger\alpha}(x)\dot{\psi}_{\alpha a}(x) - \psi_a^{\dagger\alpha}(x)\left[\psi_{\alpha a}(x), \int d^D y V(\psi^{\dagger}(y)\gamma^0\psi(y))\right]&\\ + V(\psi^{\dagger}(x)\gamma^0\psi(x)).&\end{aligned} \tag{4.11}$$

Now consider a term in the potential which takes the form $(\psi^{\dagger}(x)\gamma^0\psi(x))^n$. For this term the commutator in (4.11) can be expanded as

$$\begin{aligned}\left[\psi_{\alpha a}(x), \int d^D y(\psi^{\dagger}(y)\gamma^0\psi(y))^n\right] = n\int d^D y\,\left[\psi_{\alpha a}(x), \psi^{\dagger}(y)\gamma^0\psi(y)\right]&\\ \times\,(\psi^{\dagger}(y)\gamma^0\psi(y))^{n-1} + O(\hbar^2).&\end{aligned} \tag{4.12}$$

Using (4.1) we compute the commutator on the right side of (4.12) to find

$$\psi_a^{\dagger\alpha}(x)\left[\psi_{\alpha a}(x), \int d^D y(\psi^\dagger(y)\gamma^0\psi(y))^n\right] = n\psi_a^{\dagger\alpha}(x)\gamma^0{}_\alpha{}^\beta\psi_{\beta a}(x) \times (\psi^\dagger(x)\gamma^0\psi(x))^{n-1} + O(\hbar^2) \tag{4.13}$$

and thus deduce that the Hamiltonian (4.11) can be written

$$\mathcal{H}[\psi(x),\bar{\psi}(x)] = i\psi_a^{\dagger\alpha}(x)\dot{\psi}_{\alpha a}(x) - \psi_a^{\dagger\alpha}(x)\gamma^0{}_\alpha{}^\beta\psi_{\beta a}(x)V'(\psi^\dagger(x)\gamma^0\psi(x)) + V(\psi^\dagger(x)\gamma^0\psi(x)) + O(\hbar^2). \tag{4.14}$$

Defining $z(x) = \bar{\psi}_a^\alpha(x)\psi_{\alpha a}(x) = \psi_a^{\dagger\alpha}(x)\gamma^0{}_\alpha{}^\beta\psi(x)_{\beta a}$ we thus require that the quantity

$$V(z) - zV'(z) \tag{4.15}$$

must be bounded from below in mean field situations, where $\hbar^2$ can be neglected. More generally we see from (4.12) that it corresponds to neglecting terms involving higher derivatives of the potential, *i.e.* those of the form

$$z\frac{\partial^n V}{\partial z^n}(z) \tag{4.16}$$

with $n > 1$. We will find that these terms may be neglected in the large N limit. Furthermore, we will learn that the correct interpretation (at large N) is to treat the $z(x)$ as a real variable. At finite N we find that treating $z(x)$ as even Grassmann (and thus a potential, $V(z)$, as a finite series) yields nonsense results typical of truncations [19,20]. Thus, throughout we treat $z(x)$ as a real number and not even Grassmann as one might naively expect. Finally, the condition that (4.15) is bounded from below implies that any potential growing larger than linear for large $z(x)$, must be bounded from below.

4.3 The Legendre flow equation

The structure of Fermionic field theory leads to a Legendre flow equation that looks somewhat different to the analogous equation for the scalars (2.22). However, in common with scalar theory it can be solved exactly and analytically in the large N limit. The application of the LPA to Legendre flow equation is non-trivial and thus at finite N we concentrate on the Polchinski formalism. Ultimately we will use the large N limit of the Legendre equation to develop an understanding of the LPA to Polchinski's equation. Begin by consider the generating functional regulated by an infra-red cutoff, similar to scalar field theory,

$$Z(\bar{\zeta}, \zeta) = \int (d\bar{\psi})(d\psi) e^{-\ \bar{\psi}^{\alpha}_{a}\ \Delta^{-1\ \ \beta}_{IR\alpha}\ \psi_{\beta a} -\ S_{\Lambda}[\bar{\psi},\psi]\ +\ \bar{\psi}^{\alpha}_{a}\ \zeta_{\alpha a}\ +\ \bar{\zeta}^{\alpha}_{a}\ \psi_{\alpha a}} \tag{4.17}$$

where $\bar{\zeta}$ and ζ are the sources. We begin by deriving the necessary preliminaries to finding the flow equation. Define $Z(\bar{\zeta}, \zeta) = e^{W(\bar{\zeta}, \zeta)}$ and introduce the Legendre effective action, Γ_Λ, by

$$\Gamma_\Lambda(\bar{\psi}^c, \psi^c)\ +\ \bar{\psi}^{c\alpha}_{\ a}\ \Delta^{-1\ \ \beta}_{IR\alpha}\ \psi^c{}_{\beta a}\ =\ -W(\bar{\zeta}, \zeta)\ +\ \bar{\psi}^{c\alpha}_{\ a}\ \zeta_{\alpha a}\ +\ \bar{\zeta}^{\alpha}_{a}\ \psi^c{}_{\alpha a} \tag{4.18}$$

so that taking the classical limit, $h \to 0$, $S_\Lambda \to \Gamma_\Lambda$. Then, if we differentiate (4.18) with respect to ψ^c and $\bar{\psi}^c$ separately, we find

$$\frac{\delta \Gamma_\Lambda}{\delta \psi^c{}_{\alpha a}}\ -\ \bar{\psi}^{c\beta}_{\ a}\ \Delta^{-1\ \ \alpha}_{IR\beta}\ =\ -\ \bar{\zeta}^{\alpha}_{a} \tag{4.19}$$

and

$$\frac{\delta \Gamma_\Lambda}{\delta \bar{\psi}^{c\alpha}_{\ a}}\ +\ \Delta^{-1\ \ \beta}_{IR\alpha} \psi^c{}_{\beta a}\ =\ \zeta_{\alpha a} \tag{4.20}$$

where the classical fields are defined by

$$\bar{\psi}^{c\alpha}_{\ a}\ =\ -\ \frac{\delta W}{\delta \zeta_{\alpha a}} \tag{4.21}$$

and

$$\psi^c{}_{\alpha a}\ =\ \frac{\delta W}{\delta \bar{\zeta}^{\alpha}_{a}} \tag{4.22}$$

similar to the scalar case (2.18). Then, by differentiating (4.19) and (4.20) with respect to $\bar{\psi}^c$ or ψ^c and differentiating (4.21) and (4.22) once with respect to $\bar{\zeta}$ or ζ we can rewrite the following matrix equation

$$\begin{pmatrix} \frac{\delta \psi^c}{\delta \zeta} & \frac{\delta \psi^c}{\delta \zeta} \\ \frac{\delta \bar{\psi}^c}{\delta \bar{\zeta}} & \frac{\delta \bar{\psi}^c}{\delta \zeta} \end{pmatrix}_{ab\ \alpha}^{\qquad\beta} \begin{pmatrix} \frac{\delta \bar{\zeta}}{\delta \bar{\psi}^c} & \frac{\delta \bar{\zeta}}{\delta \bar{\psi}^c} \\ \frac{\delta \zeta}{\delta \psi^c} & \frac{\delta \zeta}{\delta \bar{\psi}^c} \end{pmatrix}_{bd\ \beta}^{\qquad\gamma} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}_{ad\ \alpha}^{\qquad\gamma} \tag{4.23}$$

as

$$\begin{pmatrix} \frac{\delta^2 W}{\delta\bar{\zeta}\delta\bar{\zeta}} & \frac{\delta^2 W}{\delta\zeta\delta\bar{\zeta}} \\ -\frac{\delta^2 W}{\delta\bar{\zeta}\delta\zeta} & -\frac{\delta^2 W}{\delta\zeta\delta\zeta} \end{pmatrix}_{ab\ \alpha}^{\quad\ \beta} \begin{pmatrix} -\frac{\delta^2\Gamma_\Lambda}{\delta\psi^c\delta\psi^c} & -\frac{\delta^2\Gamma_\Lambda}{\delta\bar{\psi}^c\delta\psi^c} + \Delta_{IR}^{-1} \\ \frac{\delta^2\Gamma_\Lambda}{\delta\psi^c\delta\bar{\psi}^c} + \Delta_{IR}^{-1} & \frac{\delta^2\Gamma_\Lambda}{\delta\bar{\psi}^c\delta\bar{\psi}^c} \end{pmatrix}_{ad\ \alpha}^{\quad\ \gamma}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}_{ad\ \alpha}^{\quad\ \gamma} \tag{4.24}$$

where I is the unit matrix

With the preliminaries complete we begin the derivation by differentiating the generating functional, (4.17), with respect to the cutoff, Λ,

$$\frac{\partial Z}{\partial\Lambda} = \frac{\delta}{\delta\zeta_{\alpha a}}\left(\frac{\partial\Delta_{IR\alpha}^{-1\ \beta}}{\partial\Lambda}\frac{\delta Z}{\delta\bar{\zeta}_a^\beta}\right) \tag{4.25}$$

or alternatively, substituting $Z = e^W$,

$$\frac{\partial W}{\partial\Lambda} = \frac{\delta W}{\delta\zeta_{\alpha a}}\frac{\partial\Delta_{IR\alpha}^{-1\ \beta}}{\partial\Lambda}\frac{\delta W}{\delta\bar{\zeta}_a^\beta} + \mathrm{tr}\left(\frac{\partial\Delta_{IR\alpha}^{-1\ \beta}}{\partial\Lambda}\frac{\delta}{\delta\zeta_{\alpha a}}\frac{\delta W}{\delta\bar{\zeta}_a^\beta}\right). \tag{4.26}$$

Then we differentiate (4.18) with respect to Λ and use (4.21) and (4.22) to find

$$\left.\frac{\partial W}{\partial\Lambda}\right|_{\zeta,\bar{\zeta}} = -\left.\frac{\partial\Gamma_\Lambda}{\partial\Lambda}\right|_{\psi^c,\bar{\psi}^c} - \bar{\psi}^{c\alpha}_{\ a}\frac{\partial\Delta_{IR\alpha}^{-1\ \beta}}{\partial\Lambda}\psi^c_{\ \beta a}. \tag{4.27}$$

Equating (4.26) and (4.27) and again using (4.21) and (4.22) we finally arrive at our flow equation

$$\frac{\partial\Gamma_\Lambda}{\partial\Lambda} = -\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & \frac{\partial\Delta_{IR}^{-1}}{\partial\Lambda} \\ \frac{\partial\Delta_{IR}^{-1}}{\partial\Lambda} & 0 \end{pmatrix}_{\alpha}^{\ \beta}\begin{pmatrix} \frac{\delta^2 W}{\delta\bar{\zeta}\delta\bar{\zeta}} & \frac{\delta^2 W}{\delta\zeta\delta\bar{\zeta}} \\ -\frac{\delta^2 W}{\delta\bar{\zeta}\delta\zeta} & -\frac{\delta^2 W}{\delta\zeta\delta\zeta} \end{pmatrix}_{aa\ \beta}^{\quad\ \alpha}\right) \tag{4.28}$$

which can be rewritten using (4.24),

$$\frac{\partial\Gamma_\Lambda}{\partial\Lambda} = -\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & \frac{\partial\Delta_{IR}^{-1}}{\partial\Lambda} \\ \frac{\partial\Delta_{IR}^{-1}}{\partial\Lambda} & 0 \end{pmatrix}_{\alpha}^{\ \beta}\right.$$

$$\left.\times\begin{pmatrix} -\frac{\delta^2\Gamma_\Lambda}{\delta\psi^c\delta\psi^c} & -\frac{\delta^2\Gamma_\Lambda}{\delta\bar{\psi}^c\delta\psi^c} + \Delta_{IR}^{-1} \\ \frac{\delta^2\Gamma_\Lambda}{\delta\psi^c\delta\bar{\psi}^c} + \Delta_{IR}^{-1} & \frac{\delta^2\Gamma_\Lambda}{\delta\bar{\psi}^c\delta\bar{\psi}^c} \end{pmatrix}^{-1\quad\alpha}_{\qquad aa\ \beta}\right) \tag{4.29}$$

which will be our starting point for solving the large N limit exactly. Compare and contrast (4.29) with (2.22).

4.4 The exact large N limit

As discussed, the Legendre flow equation is exactly solvable in the large N limit and thus we will use this limit to establish some exact results with which we will later compare the LPA (as applied to the Polchinski formalism). We begin by assuming that we can write $\Gamma_\Lambda = \Gamma_\Lambda[z(x)]$ for some Λ [23], where we have defined the scalar $z(x) = \bar{\psi}^\alpha_a(x)\psi_{\alpha a}(x)$. Then differentiating Γ_Λ with respect to the fields, we observe that the diagonal terms of (4.29) become zero and the off diagonal terms take the form

$$\delta_{ab}\Delta^{-1}_{IR\beta}{}^{\alpha} + \delta_{ab}\delta_\beta{}^\alpha\frac{\delta\Gamma_\Lambda}{\delta z} + \bar{\psi}^\alpha_a\psi_{\beta b}\frac{\delta^2\Gamma_\Lambda}{\delta z\delta z}. \tag{4.30}$$

Thus the inverse in (4.29) is of the form $(aI+b)^{-1} = a^{-1}I - a^{-1}ba^{-1} + a^{-1}ba^{-1}ba^{-1} - \cdots$ where a is a flavour singlet, I is the unit matrix in flavour space and b is an arbitrary matrix in flavour space. With the trace over flavours in (4.29) this implies that the third term in (4.30) contributes negligibly for large $N_f = \delta_{aa}$. Taking this limit, (4.29) becomes

$$\frac{\partial\Gamma_\Lambda}{\partial\Lambda} = -N_f\mathrm{tr}\left(\frac{\partial\Delta^{-1}_{IR}}{\partial\Lambda}{}_\alpha{}^\beta\left(\frac{\delta\Gamma_\Lambda}{\delta z}I + \Delta^{-1}_{IR}\right)^{-1}{}_\beta{}^\alpha\right). \tag{4.31}$$

If for some value of Λ, Γ_Λ is a functional of z only (as assumed) then $\frac{\delta\Gamma_\Lambda}{\delta z}I + \Delta^{-1}_{IR}$ is also a functional of z only, and thus, given that (4.31) is a first order differential equation in Λ, we have that this form is preserved by the flow, *i.e.* $\Gamma_\Lambda = \Gamma_\Lambda[z]$ for all Λ[2]. Furthermore, the fact that the flow equation (4.31) now contains only zero or one point functions allows for a considerable simplification [23]: Set $z(x) = z$ a constant. Then, since the effective action may be written as a derivative expansion (to all orders), we see that only a potential term of the form

$$\Gamma_\Lambda = \int d^Dx V(z,\Lambda) \tag{4.32}$$

[2]this has resulted in a restriction to a submanifold in the coupling constant space [23]

survives. Thus, substituting (4.32) into (4.31) and transforming into momentum space

$$\frac{\partial V}{\partial \Lambda} = N_f \Omega \int_0^\infty dq q^{D-1} \frac{\partial \Delta_{IR}}{\partial \Lambda}{}_\alpha{}^\beta \Delta^{-1}_{IR\beta}{}^\gamma \left(I + \Delta_{IR} \frac{\partial V}{\partial z} \right)^{-1}{}_\gamma{}^\alpha \tag{4.33}$$

where we have used $\frac{\partial \Delta^{-1}_{IR}}{\partial \Lambda} = -\Delta^{-1}_{IR} \frac{\partial \Delta_{IR}}{\partial \Lambda} \Delta^{-1}_{IR}$ and $\Omega(2\pi)^D$ is the solid angle of a $(D-1)$-sphere. Now we introduce an infra-red cutoff function, $C_{IR}(i\frac{\not{q}}{\Lambda})$, such that

$$\Delta_{IR\alpha}{}^\beta = (\alpha + i\not{q})^{-1}{}_\alpha{}^\gamma C_{IR\gamma}{}^\beta, \tag{4.34}$$

where α is an unspecified constant[3] and $\Delta_{IR\alpha}{}^\beta \sim -i\not{q}^{-1}{}_\alpha{}^\beta$ for large momentum. Note that this cutoff generally breaks the chiral symmetry (later we will choose to restrict the cutoff function and the purpose of the seemingly redundant α will become clear). Here, as with the scalars, $C_{IR} \to 1$ as $q \to \infty$ such that the physics is independent of the infra-red cutoff at scales much larger than Λ and the theory is well regulated. In terms of the cutoff function (4.33) can now be written

$$\begin{aligned} \frac{\partial V}{\partial \Lambda} &= -iN_f \Omega \int_0^\infty dq q^{D-1} \left(\frac{1}{\Lambda^2} (\alpha \not{q}^{-1} + i)^{-1}{}_\alpha{}^\beta C'_{IR\beta}{}^\gamma C^{-1}_{IR\gamma}{}^\delta (\alpha + i\not{q})_\delta{}^\epsilon \right. \\ &\times \left. \left(I + (\alpha + i\not{q})^{-1} C_{IR} \frac{\partial V}{\partial z} \right)^{-1}{}_\epsilon{}^\alpha \right). \end{aligned} \tag{4.35}$$

After scaling into dimensionless quantities, $V \to \Lambda^D V$, $z \to \Lambda^{D-1} z$, $q \to \Lambda q$ and $\alpha \to \Lambda \alpha$, and using renormalization time defined by (1.29) this becomes,

$$\begin{aligned} \frac{\partial V}{\partial t} &= DV + (1-D) z \frac{\partial V}{\partial z} + iN_f \Omega \int_0^\infty dq q^{D-1} \left((\alpha \not{q}^{-1} + i)^{-1}{}_\alpha{}^\beta C'_{IR\beta}{}^\gamma C^{-1}_{IR\gamma}{}^\delta \right. \\ &\times \left. (\alpha + i\not{q})_\delta{}^\epsilon \left(I + (\alpha + i\not{q})^{-1} C_{IR} \frac{\partial V}{\partial z} \right)^{-1}{}_\epsilon{}^\alpha \right). \end{aligned} \tag{4.36}$$

The cutoff function takes the general form,

$$C_{IR\alpha}{}^\beta = g(q^2)\delta_\alpha{}^\beta + ih(q^2)\not{q}_\alpha{}^\beta \tag{4.37}$$

[3]not to be confused with the indices

and

$$C^{-1}_{IR\alpha}{}^{\beta} = b(q^2)\delta_\alpha{}^\beta + ic(q^2)\not{q}_\alpha{}^\beta \tag{4.38}$$

with

$$b = \frac{g}{g^2 + h^2 q^2} \tag{4.39}$$

and

$$c = -\frac{h}{g^2 + h^2 q^2} \tag{4.40}$$

where b, c, g and h are arbitrary functions of the now dimensionless q^2. After some manipulation (4.36) can be written

$$\begin{aligned}
\frac{\partial V}{\partial t} &= DV + (1-D)z\frac{\partial V}{\partial z} + iN_f\Omega \int_0^\infty dq q^{D-1} \left(\frac{\alpha\not{q} - iq^2}{\alpha^2 + q^2}\delta_\alpha{}^\beta \right. \\
&\times \left((2q^2h' + h) - 2ig'\not{q}\right)_\beta{}^\gamma (b + ic\not{q})_\gamma{}^\delta \\
&\times \left. (\alpha + i\not{q})_\delta{}^\epsilon \frac{H - G\not{q}}{H^2 - q^2G^2}\delta_\epsilon{}^\alpha \right)
\end{aligned} \tag{4.41}$$

where prime denotes differentiation with respect to the argument, and

$$H(q^2) = 1 + \frac{\alpha g - hq^2}{\alpha^2 + q^2}\frac{\partial V}{\partial z} \tag{4.42}$$

and

$$G(q^2) = i\frac{\alpha h - g}{\alpha^2 + q^2}\frac{\partial V}{\partial z}. \tag{4.43}$$

We choose specifically $h = 0$ and $g = \theta_\epsilon(q, 1)$, where $\theta_\epsilon(q, 1)$ is a smooth cutoff function, satisfying $\theta_\epsilon(q, 1) \approx 0$ for $q < 1 - \epsilon$, $\theta_\epsilon(q, 1) \approx 1$ for $q > 1 + \epsilon$ and $0 < \theta_\epsilon < 1$. Ultimately we take the sharp cutoff limit, $\theta_\epsilon(q, 1) \to \theta(q - 1)$ as $\epsilon \to 0$, and the flow equation will simplify enormously. Note that the cutoff breaks the chiral symmetry unless $\alpha = 0$. Using $\not{q}_\alpha{}^\alpha = q_\mu \mathrm{tr}(\gamma^\mu) = 0$ and writing $N = N_f N_i = N_f \delta_\alpha{}^\alpha$, we arrive at

$$\frac{\partial V}{\partial t} = DV + (1-D)z\frac{\partial V}{\partial z} + 2N\Omega \int_0^\infty dq q^{D+1} \frac{Hg'}{g(H^2 - q^2G^2)}. \tag{4.44}$$

Then using (4.42) and (4.43) and the lemma presented in Appendix 4A [18] we take the $\epsilon \to 0$ limit,

$$\frac{\partial V}{\partial t} = DV + (1-D)z\frac{\partial V}{\partial z} + N\Omega\left[\ln\left(\frac{t^2}{t^2\left(\frac{\partial V}{\partial z}\right)^2 + 2\alpha t\frac{\partial V}{\partial z} + \alpha^2 + 1}\right)\right]_{t=0}^{t=1} \tag{4.45}$$

so that we finally reach the desired flow equation by a change of variables,

$$DV \to DV - N\Omega\left[\ln(t^2)\right]_{t=0}^{t=1} \tag{4.46}$$

yielding

$$\frac{\partial V}{\partial t} = DV + (1-D)z\frac{\partial V}{\partial z} - \ln\left(1 + \frac{\left(\frac{\partial V}{\partial z}\right)^2 + 2\alpha\frac{\partial V}{\partial z}}{\alpha^2+1}\right). \tag{4.47}$$

In the last step we scaled out the N and Ω dependencies via $V \to \Omega N V$ and $z \to \Omega N z$ (note that these scalings justify the neglect of terms of the form (4.16) in (4.15)). Differentiating (4.47) with respect to z and defining $W = \frac{\partial V}{\partial z}$ we find

$$\frac{\partial W}{\partial t} = W + \left((1-D)z - \frac{2(W+\alpha)}{\alpha^2 + 1 + W^2 + 2\alpha W}\right)\frac{\partial W}{\partial z}. \tag{4.48}$$

The flow equation in its forms (4.47) and (4.48) will form the focus of our attention. We realise the Gaussian solution, $W = V = 0$, but in the following will concentrate on non-trivial solutions.

4.5 Solutions in the large N limit

We now solve the large N limit [42,43] of the Legendre flow equation ((4.47) and (4.48)) for the fixed points and eigenvalue spectrum. Initially we choose $\alpha = 0$ to analyse specifically the solutions with a cutoff introduced in a chirally invariant manner, however ultimately we will allow α to be non-zero to make meaningful comparisons with the Polchinski approach to be developed later. The eigenvalues are found by perturbing the potential in (4.47). We substitute $V = V_* + \epsilon\sum_n U_n e^{\lambda_n t}$, use the fixed point version of (4.47) and expand the logarithmic term, yielding to $O(\epsilon)$

$$(\lambda_n - D)U_n = \left((D-1)z + \frac{2(W_* + \alpha)}{\alpha^2 + 1 + W_*^2 + 2\alpha W_*}\right)\frac{\partial U_n}{\partial z}. \tag{4.49}$$

Using the fixed point version of (4.48) it is straightforward to see that

$$\begin{aligned} \frac{\partial U_n}{\partial W_*} &= \frac{\partial U_n}{\partial z} \Big/ \frac{\partial W_*}{\partial z} \\ &= (\lambda_n - D)\frac{U_n}{W_*} \end{aligned} \tag{4.50}$$

and it is easily verified that

$$U_n = K_1 W_*^{D-\lambda_n}. \tag{4.51}$$

where K_1 is an arbitrary constant. Insisting that the eigenfunctions, U_n, are analytic and providing that W_* crosses the z-axis we thus deduce that the power in (4.51) must be a positive integer, n, and therefore that

$$\lambda_n = D - n \tag{4.52}$$

regardless of α. (Note, that these eigenvalues are degenerate with the Gaussian in two dimensions.)

Specialising to three dimensions the exact flow equation, in its form (4.48), can be solved analytically using the method of characteristics. The solution is found to be

$$\begin{aligned} z &= \frac{1}{\alpha^2+1}\left(\alpha(\alpha^2-3)\ln\left(\frac{W^2}{W^2+2\alpha W+\alpha^2+1}\right)\right. \\ &+ \left. 2(3\alpha^2-1)\tan^{-1}(W+\alpha)\right)W^2 + \frac{4\alpha^2 W}{(\alpha^2+1)^2} \\ &- \frac{2W+\alpha}{\alpha^2+1} + f_1(We^{-t})W^2. \end{aligned} \tag{4.53}$$

where the arbitrary function, $f_1(We^{-t})$, is set constant (denoted by f_1) for a fixed point solution. However, we find we need to impose conditions on the range of f_1 for the solution to be considered physical. We require that the solution W is defined for all z and that (4.15), and thus the negative potential, is bounded from below. After a little analysis of (4.53) we deduce the large W behaviour as

$$z \sim \frac{(3\alpha^2-1)\pi}{\alpha^2+1}|W|W + f_1 W^2, \tag{4.54}$$

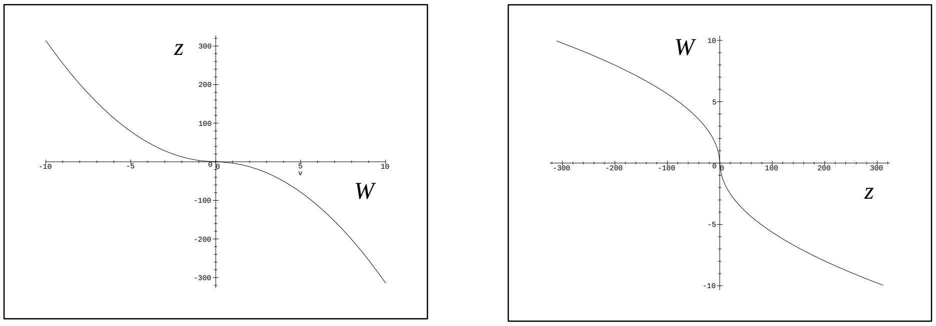

Figure 4.1: Solution to the large N limit of the Legendre flow equation in three dimensions with cutoff introduced in a chirally symmetric manner

which leads to a restriction in the range of f_1:

$$\frac{(3\alpha^2-1)\pi}{\alpha^2+1} < f_1 < \frac{(1-3\alpha^2)\pi}{\alpha^2+1}. \tag{4.55}$$

Here we have imposed that $W(z) > 0$ for large negative z and that $W(z) < 0$ for large positive z. Furthermore we demand that W is not double valued in z. This is done by simply differentiating (4.53) and studying its turning points. By experiment we find that (4.55) provides a good approximation to this restriction.
Choosing $\alpha = 0$ yields a flow equation for the cutoff introduced in such a manner as to maintain chiral symmetry which has the following solution:

$$z = -2W - 2W^2\tan^{-1}(W) + f_1 W^2. \tag{4.56}$$

A chirally symmetric potential must be even in z and thus we observe that (4.56) actually breaks this symmetry unless we set $f_1 = 0$. This solution is plotted in figure 4.1. It is noticed that independent of f_1, the solution will always satisfy $W(0) = 0$ and thus the eigenvalue spectrum is indeed given by (4.52), $\lambda_n = 3 - n$. These results are in direct agreement with those of Zinn-Justin [43]. The marginal eigenvalue, $\lambda_3 = 0$, is associated with a line of fixed points parameterised by f_1. We find that the double valued condition discussed above restricts the acceptable solutions to the range $-3.0708 < f_1 < 3.0708$.

Allowing α to be non-zero, we can study the symmetry breaking scenario. For example we could start by fixing f_1 and altering α, such that for the two boundedness conditions, (4.55),

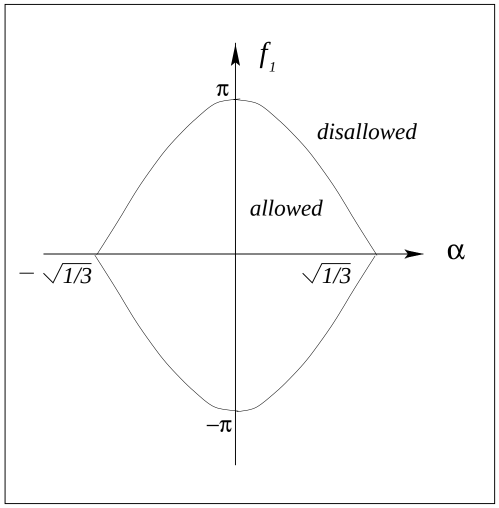

Figure 4.2: Region of $(\alpha,\ f_1)$ space allowed by the boundedness condition (4.55)

to be consistent we require $\alpha^2 < \frac{1}{3}$. Thus the region of 'parameter space' allowed is constrained (figure 4.2). However, generally the double valued condition restricts the region allowed even further. For example for $\alpha = 0.25$ (4.55) implies $-2.4024 < f_1 < 2.4024$ whereas in actual fact we find $-2.2489 < f_1 < 2.1325$. We recognise that (4.53) is unchanged if we simultaneously let $\alpha \to -\alpha$, $z \to -z$ and $W \to -W$ (this is discrete chiral symmetry) and thus focus on just $\alpha > 0$. The imposition of (4.55) and the observation that (4.53) has no divergences has the implication that all the acceptable solutions satisfy $W(z) = 0$ for some z and thus for all these fixed points the eigenvalue spectrum is again $\lambda_n = 3 - n$.

4.6 The Polchinski flow equation

Here we present a constructive proof of the Polchinski flow equation for a purely Fermionic theory. This will form the focus of our attention when we apply the LPA. Begin by considering the generating functional regulated by an overall momentum cutoff Λ_o, made explicit later. For a Fermion field $\psi(x)$ with propagator $\Delta(p)$ and arbitrary bare action $S_{\Lambda_o}[\bar{\psi}, \psi]$, we write

$$Z(\bar{\zeta}, \zeta) = \int (d\bar{\psi})(d\psi) e^{-\ \bar{\psi}^{\alpha}_{a}\ \Delta^{-1}{}_{\alpha}{}^{\beta}\ \psi_{\beta a} -\ S_{\Lambda_o}[\bar{\psi},\psi]\ +\ \bar{\psi}^{\alpha}_{a}\ \zeta_{\alpha a}\ +\ \bar{\zeta}^{\alpha}_{a}\ \psi_{\alpha a}} \tag{4.57}$$

where, as before $\bar{\zeta}$ and ζ are sources, α and β run over internal spinor indices from 1 to N_i and a is the flavour index which runs from 1 to N_f. Following the methods of [18] presented in

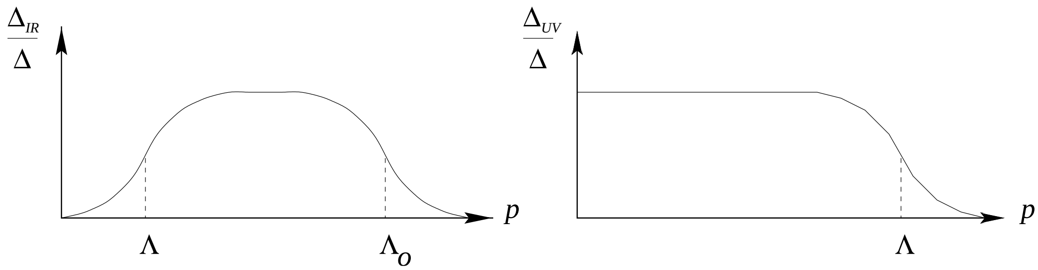

Figure 4.3: Ultra-violet and Infra-red cutoff functions

chapter two we split the propagator and field into high and low momentum parts,

$$\Delta(p) = \Delta_{IR}(p) + \Delta_{UV}(p) \tag{4.58}$$

and

$$\psi = \psi_{IR} + \psi_{UV} \tag{4.59}$$

respectively. Then we can write (4.57) as

$$Z(\bar{\zeta},\zeta) = \int (d\bar{\psi}_{UV})(d\psi_{UV}) e^{-\ \bar{\psi}_{UV}{}_a^{\alpha}\ \Delta^{-1}_{UV\alpha}{}^{\beta}\ \psi_{UV\beta a}}\ Z_\Lambda[\bar{\psi}_{UV},\psi_{UV},\bar{\zeta},\zeta] \tag{4.60}$$

where

$$\begin{aligned} Z_\Lambda[\bar{\psi}_{UV},\psi_{UV},\bar{\zeta},\zeta] = &\int (d\bar{\psi}_{IR})(d\psi_{IR}) e^{-\ \bar{\psi}_{IR}{}_a^{\alpha}\ \Delta^{-1}_{IR\alpha}{}^{\beta}\ \psi_{IR\beta a}} \\ \times\ &e^{-\ S_{\Lambda_o}[\bar{\psi}_{IR}+\bar{\psi}_{UV},\psi_{IR}+\psi_{UV}]\ +\ (\bar{\psi}_{IR}+\bar{\psi}_{UV})_a^{\alpha}\ \zeta_{\alpha a}\ +\ \bar{\zeta}_a^{\alpha}\ (\psi_{IR}+\psi_{UV})_{\alpha a}}. \end{aligned} \tag{4.61}$$

We verify (4.60) by eliminating ψ_{IR} and $\bar{\psi}_{IR}$ using (4.59) and substituting $\psi_{UV} = \psi_{UV}^{'} + \Delta_{UV}\ \Delta^{-1}\ \psi$ to retrieve (4.57) up to a multiplicative factor which we ignore. We now make the momentum cutoffs explicit by defining $\Delta_{IR} = [\theta_\epsilon(p,\Lambda) - \theta_\epsilon(p,\Lambda_o)]\Delta$ and $\Delta_{UV} = [1-\theta_\epsilon(p,\Lambda)]\Delta$ where similar to our previous analysis θ_ϵ is a smooth function satisfying $\theta_\epsilon(p,\Lambda) \approx 0$ for $p < \Lambda - \epsilon$, $\theta_\epsilon(p,\Lambda) \approx 1$ for $p > \Lambda - \epsilon$ and $0 \leq \theta_\epsilon(p,\Lambda) \leq 1$ (figure 4.3). The derivation of the flow equation starts by substituting $\psi_{IR} = \psi - \psi_{UV}$ into (4.61), yielding

$$\begin{aligned} Z_\Lambda = &\ e^{-\ \bar{\psi}_{UV}{}_a^{\alpha}\ \Delta^{-1}_{IR\alpha}{}^{\beta}\ \psi_{UV\beta a}\ -\ S_{\Lambda_o}[\frac{\delta}{\delta\bar{\zeta}},\frac{\delta}{\delta\zeta}]} \\ \times &\int (d\bar{\psi})(d\psi) e^{-\ \bar{\psi}_a^{\alpha}\ \Delta^{-1}_{IR\alpha}{}^{\beta}\ \psi_{\beta a}\ +\ \bar{\psi}_a^{\alpha}\ [\Delta^{-1}_{IR}\ \psi_{UV}+\zeta]_{\alpha a}\ +\ [\bar{\psi}_{UV}\ \Delta^{-1}_{IR}+\bar{\zeta}]_a^{\alpha}\ \psi_{\alpha a}}. \end{aligned} \tag{4.62}$$

Performing the Gaussian integral in (4.62)

$$Z_\Lambda = e^{-\ \bar\psi_{UV}{}^{\alpha}_{a}\ \Delta_{IR\alpha}^{-1}{}^{\beta}\ \psi_{UV\beta a}\ -\ S_{\Lambda_o}[\frac{\delta}{\delta\zeta},\frac{\delta}{\delta\bar\zeta}]\ +\ [\bar\psi_{UV}\ \Delta_{IR}^{-1}+\bar\zeta]^{\alpha}_{a}\ \Delta_{IR\alpha}{}^{\beta}\ [\Delta_{IR}^{-1}\ \psi_{UV}+\zeta]_{\beta a}} \tag{4.63}$$

$$= e^{\ \bar\zeta^{\alpha}_{a}\ \Delta_{IR\alpha}{}^{\beta}\ \zeta_{\beta a}\ +\ \bar\psi_{UV}{}^{\alpha}_{a}\ \zeta_{\alpha a}\ +\ \bar\zeta^{\alpha}_{a}\ \psi_{UV\alpha a}\ -\ S_\Lambda[\bar\psi_{UV}+\bar\zeta\Delta_{IR},\psi_{UV}+\Delta_{IR}\zeta]} \tag{4.64}$$

so that (4.60) becomes

$$\begin{aligned} Z(\bar\zeta,\zeta) \ &= \ \int (d\bar\psi_{UV})(d\psi_{UV}) e^{-\ \bar\psi_{UV}{}^{\alpha}_{a}\ \Delta_{UV\alpha}^{-1}{}^{\beta}\ \psi_{UV\beta a}\ +\ \bar\zeta^{\alpha}_{a}\ \Delta_{IR\alpha}{}^{\beta}\ \zeta_{\beta a}} \\ &\times\ \ e^{\ \bar\psi_{UV}{}^{\alpha}_{a}\ \zeta_{\alpha a}\ +\ \bar\zeta^{\alpha}_{a}\ \psi_{UV\alpha a}\ -\ S_\Lambda[\bar\psi_{UV}+\bar\zeta\Delta_{IR},\psi_{UV}+\Delta_{IR}\zeta]} \end{aligned} \tag{4.65}$$

for some functional, S_Λ.

Now consider that $\bar\zeta$ and ζ only couple to the low momentum modes, *i.e.* such that $\bar\zeta\Delta_{IR} = \Delta_{IR}\zeta = 0$. Then, in this limit (4.65) becomes

$$\begin{aligned} Z(\bar\zeta,\zeta) \ &= \ \int (d\bar\psi_{UV})(d\psi_{UV}) e^{-\ \bar\psi_{UV}{}^{\alpha}_{a}\ \Delta_{UV\alpha}^{-1}{}^{\beta}\ \psi_{UV\beta a}} \\ &\times\ \ e^{\ \bar\psi_{UV}{}^{\alpha}_{a}\ \zeta_{\alpha a}\ +\ \bar\zeta^{\alpha}_{a}\ \psi_{UV\alpha a}\ -\ S_\Lambda[\bar\psi_{UV},\psi_{UV}]} \end{aligned} \tag{4.66}$$

and thus S_Λ coincides with the Wilsonian effective action. Differentiating (4.61) with respect to the cutoff, Λ, gives

$$\frac{\partial Z_\Lambda}{\partial\Lambda} = \left(\frac{\delta}{\delta\zeta} + \bar\psi_{UV}\right)^{\alpha}_{a} \frac{\partial \Delta_{IR}^{-1}}{\partial\Lambda}{}_{\alpha}{}^{\beta} \left(\frac{\delta}{\delta\bar\zeta} - \psi_{UV}\right)_{\beta a} Z_\Lambda. \tag{4.67}$$

Defining $\underline{\psi}_{\alpha a} = \psi_{UV\alpha a}\ +\ \Delta_{IR\alpha}{}^{\beta}\ \zeta_{\beta a}$ and $\underline{\bar\psi}^{\alpha}_{a} = \bar\psi_{UV}{}^{\alpha}_{a}\ +\ \bar\zeta^{\beta}_{a}\ \Delta_{IR\beta}{}^{\alpha}$ and substituting (4.64) into (4.67) we arrive at our flow equation,

$$\frac{\partial S_\Lambda}{\partial\Lambda} = \frac{\delta S_\Lambda}{\delta\underline{\psi}_{\alpha a}}\ K_{\Lambda\alpha}{}^{\beta}\ \frac{\delta S_\Lambda}{\delta\underline{\bar\psi}^{\beta}_{a}}\ -\ \mathrm{tr}\left(K_{\Lambda\alpha}{}^{\beta}\ \frac{\delta^2 S_\Lambda}{\delta\underline{\psi}_{\alpha a}\delta\underline{\bar\psi}^{\beta}_{\alpha}}\right) \tag{4.68}$$

where

$$K_\Lambda = \frac{\partial\Delta_{IR}}{\partial\Lambda} = -\frac{\partial\Delta_{UV}}{\partial\Lambda}. \tag{4.69}$$

It is possible to confirm this solution by eliminating S_Λ in favour of $S_{ren}\ =\ \underline{\bar\psi}^{\alpha}_{a}\ \Delta_{UV\alpha}^{-1}{}^{\beta}\ \underline{\psi}_{\beta a} + S_\Lambda$ and then showing that $\frac{\partial Z}{\partial\Lambda} = 0$ as required. Note the similarity between (2.14) and (4.68).

4.7 Application of the Local Potential Approximation

Previous attempts to solve the flow equation [44] (4.68) have involved introducing an ultraviolet cutoff function $C_{UV}(\frac{q^2}{\Lambda^2})$, as described for the scalars. However, typically various other approximations have been made in addition to the LPA[4], leading to results of dubious reliability. Here we only require that the cutoff function be dimensionless and Lorentz invariant, and so for the Fermions we allow $C = C_{UV}$ to be a function of $\not{q}$, $C(i\frac{\not{q}}{\Lambda})$, and thus write

$$\Delta_{UV\alpha}{}^{\beta} = -iC_{\alpha}{}^{\gamma}\not{q}^{-1}{}_{\gamma}{}^{\beta}, \tag{4.70}$$

where $C(0) = 1$ and C falls to zero sufficiently quickly as $q \to \infty$, such that the theory is well regulated.

Substituting (4.70) into (4.68) and replacing $\underline{\psi}$ with ψ,

$$\frac{\partial S_\Lambda}{\partial \Lambda} = \frac{\delta S_\Lambda}{\delta \psi_{\alpha a}} \frac{C'}{\Lambda^2} \delta_\alpha{}^\beta \frac{\delta S_\Lambda}{\delta \bar{\psi}^\beta_a} - \mathrm{tr}\left(\frac{C'}{\Lambda^2} \delta_\alpha{}^\beta \frac{\delta^2 S_\Lambda}{\delta \psi_{\alpha a} \delta \bar{\psi}^\beta_a} \right) \tag{4.71}$$

where the prime denotes differentiation with respect to its argument. Now we introduce the LPA by writing

$$S_\Lambda = \int d^D x V(\bar{\psi}, \psi; \Lambda) \tag{4.72}$$

such that V does not contain any derivative terms. Similarly we restrict $C(\not{\partial})$. Within this approximation we find that (4.71) reduces to

$$\frac{\partial V}{\partial \Lambda} = \frac{1}{\Lambda^2} \frac{\partial V}{\partial \psi_{\alpha a}} C'(0) \frac{\partial V}{\partial \bar{\psi}^\alpha_a} - \tau \Lambda^{D-2} \frac{\partial^2 V}{\partial \psi_{\alpha a} \partial \bar{\psi}^\alpha_a}, \tag{4.73}$$

where $I\tau = \Lambda^{-D}(2\pi)^{-D} \int d^D q C'(\not{q}/\Lambda)$ for a dimensionless constant τ and the $C'(0)$ has been transformed into configuration space. By scaling $V \to V\Lambda^D$, $\psi \to \psi\Lambda^{\frac{1}{2}(D-1)}$, $\bar{\psi} \to \bar{\psi}\Lambda^{\frac{1}{2}(D-1)}$, defining the renormalization time by (1.29) and finally scaling out the remaining cutoff function dependencies we derive the following scheme independent flow equation:

$$\frac{\partial V}{\partial t} = DV - \frac{1}{2}(D-1)\left(\psi_{\alpha a} \frac{\partial V}{\partial \psi_{\alpha a}} + \bar{\psi}^\alpha_a \frac{\partial V}{\partial \bar{\psi}^\alpha_a} \right) + \frac{\partial V}{\partial \psi_{\alpha a}} \frac{\partial V}{\partial \bar{\psi}^\alpha_a} - \frac{\partial^2 V}{\partial \psi_{\alpha a} \partial \bar{\psi}^\alpha_a} \tag{4.74}$$

[4] most notably truncations

where all quantities are now dimensionless. Note the striking similarity between this and the scalar case, (2.33). The Gaussian and High Temperature solutions are given by

$$V_*^G = 0 \tag{4.75}$$

and

$$V_*^{HT} = \frac{N}{D} + \bar{\psi}_a^\alpha \psi_{\alpha a} \tag{4.76}$$

respectively.

By perturbing about a fixed point solution, $V(\bar{\psi}, \psi, t) = V_*(\bar{\psi}, \psi) + \epsilon \sum_n f_n(\bar{\psi}, \psi) e^{\lambda_n t}$, where V_* is the fixed point under consideration and ϵ is a small parameter, we easily find an equation for the perturbations around a fixed point (ignoring $O(\epsilon^2)$),

$$\begin{aligned} \lambda_n f_n = D f_n - \frac{1}{2}(D-1)\left(\psi_{\alpha a}\frac{\partial f_n}{\partial \psi_{\alpha a}} + \bar{\psi}_a^\alpha \frac{\partial f_n}{\partial \bar{\psi}_a^\alpha}\right) \\ + \frac{\partial f_n}{\partial \psi_{\alpha a}}\frac{\partial V_*}{\partial \bar{\psi}_a^\alpha} + \frac{\partial V_*}{\partial \psi_{\alpha a}}\frac{\partial f_n}{\partial \bar{\psi}_a^\alpha} - \frac{\partial^2 f_n}{\partial \psi_{\alpha a}\partial \bar{\psi}_i^a}. \end{aligned} \tag{4.77}$$

This can then be used to find the exponents which are related to the λ_n in the familiar way (section 2.4). As for the scalars, the eigenvalues for the Gaussian fixed point are given by the mass dimension of the couplings:

$$\lambda_n^G = D - n(D-1) \tag{4.78}$$

where n is a positive integer. The eigenvalues for the High Temperature fixed point are independent of N,

$$\lambda_n^{HT} = D - n(D+1), \tag{4.79}$$

as expected for an infinitely massive theory (seen by calculating the β function for the mass, similar to that described for scalar field theory).

4.8 The large N limit of the LPA

We now consider the large N limit of the LPA Polchinski flow equation (4.74) and compare it to the exact results presented in section 4.5. Similar to before we define $z(x) = \bar{\psi}^{\alpha}_{a}(x)\psi_{\alpha a}(x)$, and consider $V = V(z,t)$. Then the flow equation becomes

$$\frac{\partial V}{\partial t} = DV - ((D-1)z + N)\frac{\partial V}{\partial z} - z\left(\frac{\partial V}{\partial z}\right)^2 - z\frac{\partial^2 V}{\partial z^2}. \tag{4.80}$$

Scaling out the N dependence via $V \to NV$ and $z \to Nz$ and taking the limit, the double derivative on the right of (4.80) vanishes. We retrieve the trivial fixed points $V^{G}_{*} = 0$ and $V^{HT}_{*} = \frac{1}{D} + z$. Then differentiating (4.80) with respect to z,

$$\frac{\partial W}{\partial t} = W - W^2 + ((1-D)z - 2zW - 1)\frac{\partial W}{\partial z} \tag{4.81}$$

where $W = \frac{\partial V}{\partial z}$.

As before we expand the potential in (4.80) as $V = V_* + \epsilon \sum_n U_n e^{\lambda_n t}$ for small ϵ. The usual manipulation leads to an equation for the eigenfunctions

$$\frac{1}{U_n}\frac{\partial U_n}{\partial z} = \frac{\lambda_n - D}{(1-D)z - 2zW_* - 1}. \tag{4.82}$$

Substituting the trivial solutions into (4.82) we rapidly reproduce the eigenvalue spectrum for the Gaussian and High Temperature fixed points. We rewrite (4.82) as

$$\begin{aligned} \frac{\partial \ln(U_n)}{\partial W_*} &= \frac{\lambda_n - D}{\frac{\partial W_*}{\partial z}((1-D)z - 2zW_* - 1)} && (4.83) \\ &= \frac{\lambda_n - D}{W_* - W_*^2} && (4.84) \end{aligned}$$

where we have used the fixed point version of (4.81). Integrating (4.84) we solve for U_n in terms of W_*,

$$U_n = K_2\left(\frac{W_* - 1}{W_*}\right)^{D-\lambda_n}, \tag{4.85}$$

where K_2 is a constant of integration. Thus we deduce that if $W_* = 1$ for some z then for U_n to be analytic at that z, the exponent in (4.85) must be a positive integer (n), implying

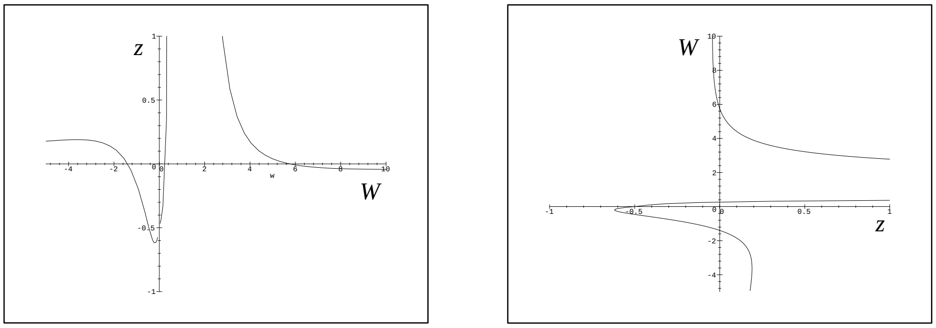

Figure 4.4: Solution to the large N limit of the LPA Polchinski flow equation in three dimensions

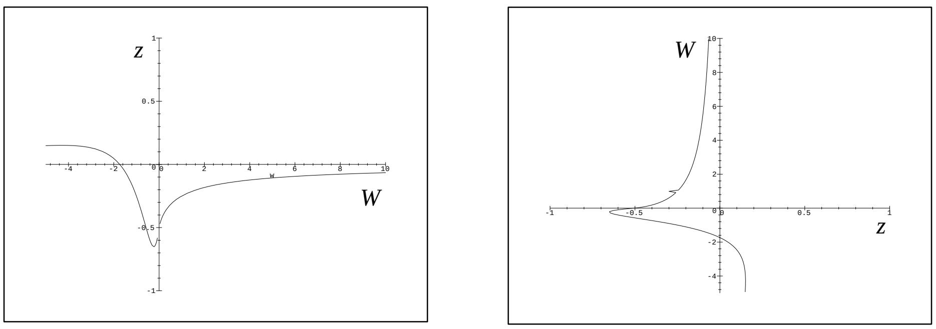

Figure 4.5: Non-divergent solution to the large N limit of the LPA Polchinski flow equation in three dimensions

$\lambda_n = D - n$, whereas if $W_* = 0$ for some z we would have $\lambda_n = D + n$.

Again, solving using the method of characteristics we find the solution in three dimensions to be

$$z = \frac{1}{(W-1)^4}\left(\frac{3}{2}W^2\ln(W^2) - W^3 + 3W - \frac{1}{2} + W^2 f_2\left(\frac{We^{-t}}{W-1}\right)\right) \tag{4.86}$$

where $f_2(\frac{We^{-t}}{W-1})$ is an arbitrary function. For a fixed point solution $f_2(\frac{We^{-t}}{W-1})$ must be set to a constant (f_2). The $f_2 = 0$ solution is shown in figure 4.4. However, from (4.86) we see that for large z, $z \sim -\frac{1}{W}$ and furthermore that we have exactly one divergence at $W = 1$, unless $f_2 = -\frac{3}{2}$. Therefore by continuity, we deduce that none of these fixed points can represent acceptable fixed points[5]. The remaining possibility, $f_2 = -\frac{3}{2}$ is plotted in figure 4.5 and is discarded for similar reasons. Hence there are no acceptable non-trivial fixed points in three dimensions.

[5]The reader can convince himself or herself of this by graphical analysis

4.9 Finite N

To study finite N theories we use (4.80). We start by substituting a series ansatz, truncated at some finite power of z. Generally we find solutions degenerate with the Gaussian at $D = 2$. However, similar to the case with scalar field theory we quickly generate large numbers of spurious solutions. The simplest ansatz yields the following solution,

$$V_*^{NT} = \frac{2-D}{2D} + \left(\frac{1}{2} - \frac{D}{4}\right) z + \left(\frac{1}{8} - \frac{D^2}{32}\right) z^2, \tag{4.87}$$

which also collapses to the High Temperature fixed point in $D = -2$, in a manner reminiscent of scalar field theory. However, as discussed we treat $z(x)$ as a real number and not as an even Grassmann and therefore we solve (4.80) numerically.

However, solving numerically, we find that non-trivial fixed points entirely disappear, unlike with scalar field theory. As discussed in section 2.7 to solve for a fixed point we simply set $\frac{\partial V}{\partial t} = 0$ and integrate (4.80) out from a given $V(0)$ until we encounter a divergence at $z = z_c$. We then invoke Griffiths analyticity[6] and look for the finite number of non-divergent solutions [13], *i.e.* ones for which $z_c \to \infty$. However, due to the fact that the cutoff function is not chirally invariant, V cannot be assumed even in z. Thus we integrate out for both positive and negative z. Figure 4.6 shows the results for three dimensional four component theory. For $z > 0$ and $V(0) < 0$ we find that the solution to (4.80) quickly diverges whereas all solutions with $V(0) > 0$ are non-divergent and seemingly acceptable fixed points. However, as discussed, we have to integrate out (4.80) for negative z. In this case the solutions diverge for all $V(0)$ with the exception of the Gaussian and High Temperature fixed points indicated. These qualitative features do not change with N. Thus we conclude that consistent with the large N limit of the LPA to Polchinski's equation we find no acceptable non-trivial fixed points in three dimensions. New divergences in z_c are generated for $z < 0$ at the Gaussian critical dimensions, those for which the λ_n^G, given by (4.78), are marginal:

$$D = \frac{n+1}{n} \tag{4.88}$$

[6] and analyticity at the origin

where $n = 1, 2, \cdots$. Thus, for $D < 2$ we find a new divergence in z_c as can be seen from figure 4.7. This divergence is degenerate with the Gaussian in exactly two dimensions, but moves away from the Gaussian as we lower the dimension. It is straightforward to see that this does not correspond to a fixed point as this solution diverges in the positive z direction. However for $D < \frac{3}{2}$ the divergence occurs for $V(0) > 0$ and corresponds to an acceptable fixed point, labelled NTFP

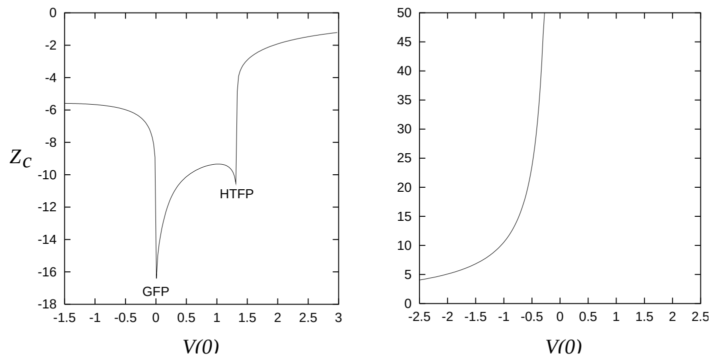

Figure 4.6: z_c versus $V(0)$ for the Fermionic Polchinski flow equation at $N = 4$ and $D = 3$

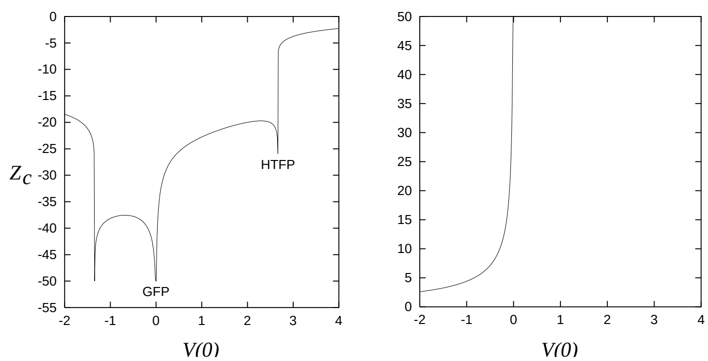

Figure 4.7: z_c versus $V(0)$ for the Fermionic Polchinski flow equation at $N = 4$ and $D = \frac{3}{2}$

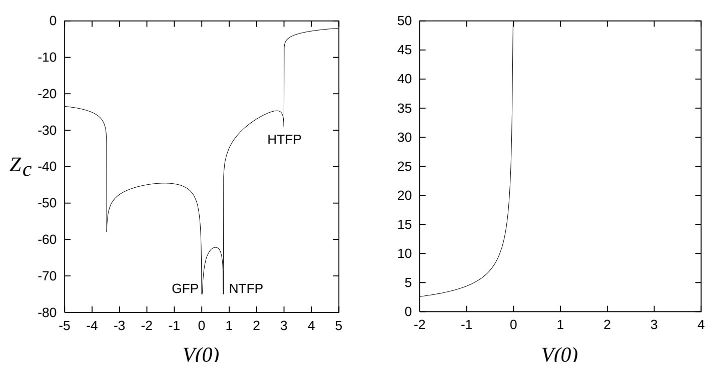

Figure 4.8: z_c versus $V(0)$ for the Fermionic Polchinski flow equation at $N = 4$ and $D = \frac{4}{3}$

in figure 4.8. This fixed point becomes degenerate with the High Temperature at $D = 1$, analogous to the first subleading fixed point for scalar field theory.

4.10 Discussion

For the Polchinski flow equation, we are at liberty to introduce the cutoff in some chirally symmetric manner, for example by insisting that C_{UV} is a function of $\frac{q^2}{\Lambda^2}$ [44]. However, if we then multiply C^{-1}_{UV} by the kinetic term in the conventional manner, we quickly realise that the right side of the exact flow equation (4.68) always carries a $\not{q}$, and thus, is trivial within the Local Potential Approximation. This problem has been alleviated here by allowing $C_{UV} = C_{UV}(\frac{\not{q}}{\Lambda})$, although it should be noted that the non-trivial LPA to Polchinski's equation can be derived, albeit less aesthetically, whilst keeping C_{UV} a function of $\frac{q^2}{\Lambda^2}$. However, in this case, the inverse cutoff function must be introduced in some non-conventional manner, such as multiplying $\frac{q^2}{\Lambda}$, which is then summed with the kinetic term[7]. Hence we might introduce the cutoff as

$$\bar{\psi}\left(\not{q} + \frac{C^{-1}_{UV}\left(\frac{q^2}{\Lambda^2}\right)}{\Lambda}q^2\right)\psi, \tag{4.89}$$

which is still afflicted with a chirally non-invariant term. Thus, in order to obtain a non-trivial LPA version of (4.68) we are essentially forced to break chiral symmetry with the cutoff function. To date, there is no known method of including momentum dependent terms within an expansion scheme for the Polchinski flow equation, without breaking reparameterisation invariance [15].

Exact analysis in the large N limit shows that only a small range of symmetry breaking cut-offs reproduce the non-trivial fixed points with the eigenvalue spectrum associated with the symmetry preserving cutoff. Thus it is not surprising that non-trivial fixed points are not seen in the LPA to the Polchinski flow equation. Generally, we interpret the results obtained from (4.80) and (4.81) as being analogous to letting α become large in (4.47) and (4.48) such that the chirally invariant component in the latter case is not dominant. From figure 4.2 the line of fixed points associated with the large N limit is seen to vanish for $\alpha^2 > \frac{1}{3}$. With this interpretation the results are seen to be entirely consistent. Thus, it has to be concluded that the hope inspired by the successes of scalar field theory discussed in chapters two and three are not yet fulfilled within Fermionic field theories.

[7] such that the right hand side of the Polchinski equation contains derivative independent terms

Appendices

Appendix 4A: Lemma

We present the lemma [18] used to (4.45):

$$\delta_\epsilon f(\theta(p,\Lambda),\Lambda) \to \delta(\Lambda - p)\int_0^1 dt f(t,p) \tag{4.90}$$

as $\epsilon \to 0$, where $f(\theta_\epsilon,\Lambda)$ is any function whose dependence on the second argument (Λ) remains continuous at $\Lambda = p$ in the limit $\epsilon = 0$. This follows from the identity

$$\delta_\epsilon f(\theta_\epsilon(p,\Lambda),\Lambda) = \left\{ \frac{\partial}{\partial\Lambda}\int_{\theta_\epsilon(p,\Lambda)}^1 dt f(t,\Lambda') \right\}\Bigg|_{\Lambda'=\Lambda} \tag{4.91}$$

where here $\delta_\epsilon(p,\Lambda) = -\frac{\partial}{\partial\Lambda}\theta_\epsilon(p,\Lambda)$. This can be seen by noting that as $\epsilon \to 0$ the lower limit on the integral becomes 1 for $p > \Lambda$ and the integral vanishes, whereas it becomes 0 for $p < \Lambda$. Thus the integral becomes a step function in Λ but with height $\int_0^1 dt f(t,\Lambda')$. Thus differentiating with respect to Λ reproduces (4.90).

Chapter 5

Concluding remarks

5.1 The Local Potential Approximation

In the context of scalar field theory, there is little doubt that the Local Potential Approximation yields reliable results, comparable with other leading methods (table 2.1). However, we have found that the additional approximation of truncations, whereby the potential is expanded as a series in the field, ϕ, and terminated at some arbitrary finite power is unreliable. In contrast, standard techniques which involve no further approximation, such as shooting, are found to be excellent methods of generating the physical quantities of interest, *e.g.* critical exponents. The new variational methods presented here provide an equally reliable alternative, with the advantage that we expect approximate solutions of LPA's for more than one invariant to be found with relative ease. Furthermore, the C-function defined using the LPA to Polchinski's flow equation provides additional justification and support for this approximation scheme.

In the case of Fermionic field theory the Legendre flow equation is significantly more difficult to deal with than that for scalar field theory, at least at finite N. However, within the LPA, the Polchinski flow equation is very similar to the corresponding scalar case. Thus, it is unfortunate that the LPA to the Polchinski equation fails to produce non-trivial fixed points for Fermionic field theory, due to our cutoff function breaking chiral symmetry. It should be considered whether re-introduction of scalars might lead to non-trivial fixed points once more.

The generalisation of (2.33) and (4.74) follows immediately,

$$\begin{aligned}\frac{\partial V}{\partial t} &= DV - \frac{1}{2}(D-2)\phi_i \frac{\partial V}{\partial \phi_i} - \left(\frac{\partial V}{\partial \phi_i}\right)^2 + \frac{\partial^2 V}{\partial \phi_i^2} \\ &- \frac{1}{2}(D-1)\left(\psi_{\alpha a}\frac{\partial V}{\partial \psi_{\alpha a}} + \bar{\psi}^\alpha_a \frac{\partial V}{\partial \bar{\psi}^\alpha_a}\right) + \frac{\partial V}{\partial \psi_{\alpha a}}\frac{\partial V}{\partial \bar{\psi}^\alpha_a} - \frac{\partial^2 V}{\partial \psi_{\alpha a}\partial \bar{\psi}^\alpha_a}\end{aligned} \tag{5.1}$$

where i is the flavour index for the scalar fields and the notation used for the Fermion fields is consistent with chapter four. However the work presented here indicates that ultimately it might be necessary to focus on other flow equations, such as the Legendre equation.

5.2 Remaining issues

Given that the LPA has been found to be competitive, both in terms of simplicity of implementation and the numerical estimates of the exponents, at least for scalar field theories, it is hoped that ultimately Fermionic field theories will pose nothing more than a problem of technique. An interesting question which we have not answered is whether there exists some method of extracting results relevant to chirally invariant theories from those presented here. We have recently initiated some interesting work analogous to that found in ref. [45] which is due to be published alongside the work presented in chapter four.

However, one of the major remaining obstacles is to deal with gauge theories. It is found that introducing a momentum cutoff destroys gauge symmetry and ultimately leads to flow equations with gauge dependent parameters. Clearly, it is difficult to make reliable physical predictions when ultimately they are gauge dependent. The most desirable solution would be to introduce the cutoff in some gauge independent way. These problems have been addressed by many authors and recently, significant progress has been made [46].

5.3 Alternative methods

A variety of other techniques with which we can deal with the renormalization group have been developed by a host of other authors. Some of the corresponding numerical estimates are compared with those due to the Local Potential Approximation in section 2.11. Although a detailed analysis of these methods is beyond the scope of this thesis, for completeness we include some references to the ϵ-expansion [1,16,42], lattice methods [47-49], perturbation theory [50-53] and large N expansions [1,42,43,54].

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